22 TOPOLOGICAL METHODS IN DISCRETE GEOMETRY

Rade T. Živaljević

INTRODUCTION

A problem is solved or some other goal achieved by "topological methods" if in our arguments we appeal to the "form," the "shape," the "global" rather than "local" structure of the object or configuration space associated with the phenomenon we are interested in. This configuration space is typically a manifold or a simplicial complex. The global properties of the configuration space are usually expressed in terms of its homology and homotopy groups, which capture the idea of the higher (dis)connectivity of a geometric object and to some extent provide "an analysis properly geometric or linear that expresses location directly as algebra expresses magnitude."[1]

Thesis: *Any global effect that depends on the object as a whole and that cannot be localized is of homological nature, and should be amenable to topological methods.*

HOW IS TOPOLOGY APPLIED IN DISCRETE GEOMETRIC PROBLEMS?

In this chapter we put some emphasis on the role of *equivariant* topological methods in solving combinatorial or discrete geometric problems that have proven to be of relevance for computational geometry and topological combinatorics and with some impact on computational mathematics in general. The versatile *configuration space/test map* scheme (CS/TM) was developed in numerous research papers over the years and formally codified in [Živ98]. Its essential features are summarized as follows:

CS/TM-1: The problem is rephrased in topological terms.

The problem should give us a clue how to define a "natural" *configuration space* X and how to rephrase the question in terms of zeros or coincidences of the associated *test maps*. Alternatively the problem may be divided into several subproblems, in which case one is often led to the question of when the solution subsets of X corresponding to the various subproblems have *nonempty intersection.*

CS/TM-2: Standard topological techniques are used in the solution of the rephrased problem.

The topological technique that is most frequently and consistently used in problems of discrete geometry is based on various forms of *generalized Borsuk-Ulam theorems.* However many other tools (Lusternik-Schnirelmann category, cup product, cup-length, intersection homology, etc.) have also found important applications.

[1] A dream of G.W. Leibniz expressed in a letter to C. Huygens dated 1697; see [Bre93, Chap. 7].

22.1 THE CONFIGURATION SPACE/TEST MAP PARADIGM

GLOSSARY

Configuration space/test map scheme (CS/TM): A very useful and general scheme for proving combinatorial or geometric facts. The problem is reduced to the question of showing that there does not exist a G-equivariant map $f : X \to V \setminus Z$ (Section 22.5) where X is the configuration space, V the test space, and Z the test subspace associated with the problem, while G is a naturally arising group of symmetries.

Configuration space: In general, any topological space X that parameterizes a class of configurations of geometric objects (e.g., arrangements of points, lines, fans, flags, etc.) or combinatorial structures (trees, graphs, partitions, etc.). Given a problem $\mathcal{P}$, an associated configuration or ***candidate space*** $X_{\mathcal{P}}$ collects all geometric configurations that are (reasonable) candidates for a solution of $\mathcal{P}$.

Test map and ***test space:*** A map $t : X_{\mathcal{P}} \to V$ from the configuration space $X_{\mathcal{P}}$ into the so-called test space V that tests the validity of a candidate $p \in X_{\mathcal{P}}$ as a solution of $\mathcal{P}$. This is achieved by the introduction of a ***test subspace*** $Z \subset V$, where $p \in X$ is a solution to the problem if and only if $t(p) \in Z$. Usually $V \cong \mathbb{R}^d$ while Z is just the origin $\{0\} \subset V$ or more generally a linear subspace arrangement in V.

Equivariant map: The final ingredient in the CS/TM-scheme is a group G of symmetries that acts on both the configuration space $X_{\mathcal{P}}$ and the test space V (keeping the test subspace Z invariant). The test map t is always assumed G-equivariant, i.e., $t(g \cdot x) = g \cdot t(x)$ for each $g \in G$ and $x \in X_{\mathcal{P}}$. Some of the methods and tools of equivariant topology are outlined in Section 22.5.

EXAMPLE 22.1.1

Suppose that ρ is a metric on $\mathbb{R}^2$ that induces the same topology as the usual Euclidean metric. Let $\Gamma \subset \mathbb{R}^2$ be a compact subspace, for example Γ can be a finite union of arcs. The problem is to find equilateral triangles in Γ, i.e. the triples (a, b, c) of distinct points in Γ such that $\rho(a, b) = \rho(b, c) = \rho(c, a)$.

This is our first example that illustrates the CS/TM-scheme. The configuration space X should collect all candidates for the solution, so a first, "naive" choice is the space of all (ordered) triples $(x, y, z) \in \Gamma$. Of course we can immediately rule out some obvious nonsolutions, e.g., degenerate triangles (x, y, z) such that at least one of numbers $\rho(x, y), \rho(y, z), \rho(z, x)$ is zero. (This illustrates the fact that in general there may be several possible choices for a configuration space associated to the initial problem.) Our choice is $X := \Gamma^3 \setminus \Delta$ where $\Delta := \{(x, x, x) \mid x \in \Gamma\}$. A "triangle" $(x, y, z) \in X$ is ρ-equilateral if and only if $(\rho(x, y), \rho(y, z), \rho(z, x)) \in Z$, where $Z := \{(u, u, u) \in \mathbb{R}^3 \mid u \in \mathbb{R}\}$. Hence a test map $t : X \to \mathbb{R}^3$ is defined by $t(x, y, z) = (\rho(x, y), \rho(y, z), \rho(z, x))$, the test space is $V = \mathbb{R}^3$, and $Z \subset \mathbb{R}^3$ is the associated test subspace. A triangle $\{x, y, z\}$, viewed as a set of vertices, is in general labelled by six different triples in the configuration space X. This redundancy is a motivation for introducing the group of symmetries $G = S_3$, which acts on both the configuration space X and the test space V. The test map t is clearly S_3-equivariant. Summarizing, we observe that the set of all equilateral triangles $\{x, y, z\}$ in Γ is in one-to-one correspondence with the set $t^{-1}(Z)/S_3$ of all S_3-orbits of all elements in $t^{-1}(Z)$. If we are interested only in the existence of (non-degenerate) equilateral triangles in Γ, it is sufficient to show that $f^{-1}(Z) \neq \emptyset$ for each S_3-equivariant map $f : X \to \mathbb{R}^3$. This is a topological problem which may be reduced to the non-existence of a S_3-equivariant map $g : X \to S^1$, where $S^1 \subset Z^\perp$ is the (S_3-invariant) unit circle in the orthogonal complement to $Z \subset \mathbb{R}^3$.

Here is another example of how topology comes into play and proves useful in geometric and combinatorial problems. The *configuration space* associated to the next problem is a 2-dimensional torus $T^2 \cong S^1 \times S^1$. This time, however, the test map is not explicitly given. Instead, the problem is reduced to counting intersection points of two "test subspaces" in T^2.

EXAMPLE 22.1.2 *A watch with two equal hands*

A watch was manufactured with a defect so that both hands (minute and hour) are identical. Otherwise the watch works well and the question is to determine the number of ambiguous positions, i.e., the positions for which it is not possible to determine the exact time.

Each position of a hand is determined by an angle $\omega \in [0, 2\pi]$, so the configuration space of all positions of a hand is homeomorphic to the unit circle S^1. Two independent hands have the 2-dimensional torus $T^2 \cong S^1 \times S^1$ as their configuration space. If θ corresponds to the minute hand and ω is the coordinate of the hour hand, then the fact that the first hand is twelve times faster is recorded by the equation $\theta = 12\,\omega$. If the hands change places then the corresponding curve Γ_2 has equation $\omega = 12\,\theta$. The ambiguous positions are exactly the intersection points of these two curves (except those that belong to the diagonal $\Delta := \{(\theta, \omega) \,|\, \theta = \omega\}$, when it is still possible to tell the exact time without knowing which hand is for hours and which for minutes). The reader can now easily find the number of these intersection points and compute that there are 143 of them in the intersection $\Gamma_1 \cap \Gamma_2$, and 11 in the intersection $\Gamma_1 \cap \Gamma_2 \cap \Delta$, which shows that there are all together 132 ambiguous positions.

REMARK 22.1.3

Let us note that the "watch with equal hands" problem reduces to counting points or 0-dimensional manifolds in the intersection of two circles, viewed as 1-dimensional submanifolds of the 2-dimensional manifold T^2. More generally, one may be interested in how many points there are in the intersection of two or more submanifolds of a higher-dimensional ambient manifold. Topology gives us a versatile tool for computing this and much more, in terms of the so-called *intersection product* $\alpha \frown \beta$ *of homology classes* α *and* β in a manifold M. This intersection product is, via Poincaré duality, equivalent to the "cup" product, and has the usual properties [Bre93, Mun84]. In our Example 22.1.2, keeping in mind that $a \frown b = -\, b \frown a$ for all 1-dimensional classes, and in particular that $a \frown a = 0$ if dim $(a) = 1$, we have $[\Gamma_1] \frown [\Gamma_2] = ([\theta] + 12[\omega]) \frown ([\omega] + 12[\theta]) = [\theta] \frown [\omega] + 12([\omega] \frown [\omega]) + 12([\theta] \frown [\theta]) + 144([\omega] \frown [\theta]) = 143([\omega] \frown [\theta])$ and, taking the orientation into account, we conclude that the number of intersection points is 143.

CONFIGURATION SPACES

The selection of an appropriate configuration space is very often the crux of the application of the CS/TM-scheme. Their construction is often based on a variety of combinatorial and geometric ideas and the following examples serve as an illustration of fairly complex configuration spaces that have appeared in actual applications.

EXAMPLE 22.1.4 *Alon's 'spaces of partitions'*

The proof of the necklace-splitting theorem (Theorem 22.4.3) provides a nice example of an application of the CS/TM scheme. A continuous model of a necklace is an interval $[0, 1]$ together with k measurable subsets $A_1, \dots, A_k$ representing "beads" of different colors. It is elementary that the configuration space of all sequences $0 \leq x_1 \leq \dots \leq x_m \leq 1$ is an m-dimensional simplex, hence the totality of all m-cuts of a necklace is also identified as an m-dimensional simplex Δ^m. Given a cut $c \in \Delta^m$, the assembling of the resulting subintervals $I_0(c), \dots, I_m(c)$ of $[0, 1]$ into r collections is determined by a function $f : [m + 1] \to [r]$. Hence, a configuration space associated to the necklace-splitting problem is obtained by gluing together m-dimensional simplices Δ^m_f, one for each function $f \in \mathrm{Fun}([m + 1], [r])$. The complex $\mathcal{N}_{m,r}$ obtained by this construction turns out to be an example of a complex obtained from a simplex by a *deleted join operation* [Mat08, Živ98].

EXAMPLE 22.1.5 *Multidimensional necklaces*

A model of a d-dimensional 'necklace', used in the proof [LŽ08] of a multidimensional version of Theorem 22.4.3 is the cube $I^d = [0, 1]^d$ together with a collection of k measurable subsets $A_1, \dots, A_k$ of I^d. The space of all partitions of I^d by axes-aligned hyperplanes (using m_i hyperplanes in the direction $i \in \{1, \dots, d\}$) is the product of simplices $Q = \Delta^{m_1} \times \dots \times \Delta^{m_d}$. A point c in the interior of Q describes a dissection of Q into d-dimensional parallelepipeds enumerated by the set $\Pi = [m_1 + 1] \times \dots \times [m_d + 1]$. The configuration space Ω_Q associated to the d-dimensional 'necklace-splitting problem' is obtained by gluing together polytopes $Q_f \cong Q$, one for each allocating function $f \in \mathrm{Fun}(\Pi, [r])$. There is an associated 'moment map' $\mu : \Omega_Q \to Q$ which makes the configuration space Ω_Q a relative of *small covers, toric spaces, (generalized) moment-angle complexes* and other objects that appear in *toric topology* [BP15] .

EXAMPLE 22.1.6 *Gromov's 'spaces of partitions'*

Very interesting polyhedral partitions are introduced by Gromov in [Gro03]. His *spaces of partitions* [Gro03, Section 5] are defined as the configuration spaces of labelled binary trees T_d of height d, with $2^d - 1$ internal nodes N_d and 2^d external nodes L_d (leaves of the tree T_d). More explicitly a labelled binary tree $(T_d, \{H_\nu\}_{\nu \in N_d})$ has an oriented hyperplane H_ν associated to each of the internal nodes $\nu \in N_d$ of T_d. The left (respectively right) outgoing edge, emanating from $\nu \in N_d$ is associated the positive half-space H^+_ν (respectively the negative half-space H^-_ν) determined by H_ν.

Each of the leaves $\lambda \in L_d$ is the end point of the unique maximal path π_λ in the tree T_d. Each of the maximal paths π_λ is associated a polyhedral region Q_λ defined as the intersection of all half-spaces associated to edges of the path

π_λ. The associated partition $\{Q_\lambda\}_{\lambda\in L_d}$ depends continuously on the chosen labels (hyperplanes) and defines an element of the associated 'space of partitions'.

These and related configuration spaces were used in [Gro03] for a proof of a general Borsuk-Ulam type theorem ($c_\bullet$-Corollary 5.3 on page 188) and utilized by Gromov for his proof of the *Waist of the Sphere Theorem.*

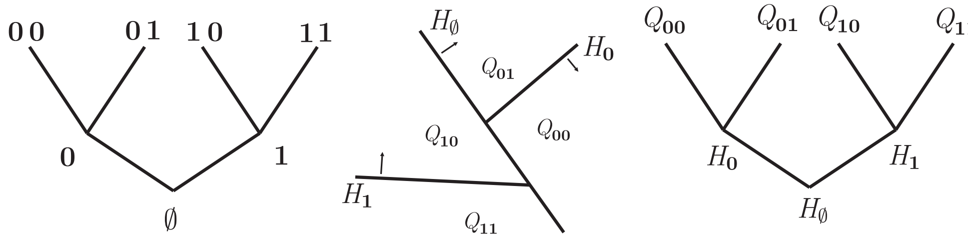

FIGURE 22.1.1
Labelled binary tree and the associated convex partition.

EXAMPLE 22.1.7 *Partitions via function-separating diagrams*

The classical configuration space $F_n(\mathbb{R}^d) = \{x \in (\mathbb{R}^d)^n \mid x_i \neq x_j \text{ for } i \neq j\}$ of labelled, distinct points in $\mathbb{R}^d$ has been used [KHA14] as a basis for constructions of convex (polyhedral) partitions of $\mathbb{R}^d$. These partitions typically arise as *generalized Voronoi partitions* (power diagrams), or more generally as 'function-separating diagrams' of finite families $\mathcal{F} = \{f_1, \dots, f_n\}$ of real-valued functions with the domain $X \subset \mathbb{R}^d$. By definition the separating diagram of $\mathcal{F}$ is the collection of sets $\{X_i\}_{i=1}^n$ where $x \in X_i$ if and only if $f_i(x) = max_{j=1,\dots,n}\{f_j(x)\}$. For a point $z = (z_1, \dots, z_n) \in F_n(\mathbb{R}^d)$ and a weight vector $(r_i) \in \mathbb{R}^n$, the associated (generalized) Voronoi partition is the function-separating diagram of the family $\{\phi_i\}_{i=1}^n$, where $\phi_i(x) = r_i - \|x - z_i\|^2$. Note that essentially the same class of convex partitions is obtained if one uses linear (affine) functions defined on $\mathbb{R}^d$.

EXAMPLE 22.1.8 *Chessboard complexes and the chessboard transform*

The chessboard transform is a procedure of constructing a simplicial complex (a generalized chessboard complex) collecting all feasible candidates for the solution of a particular Tverberg-type problem (Section 22.4). Suppose that $S = \{x_i\}_{i=1}^m \subset \mathbb{R}^d$ is a finite set ('cloud') of points of cardinality m. This set may be structured or enriched by some (unspecified) structure $\mathcal{S}$ on S, for example the points may be colored in k-colors by a coloring map $\phi : [m] \to [k]$ where the cardinality m_i of each monochromatic subsets $C_i = \phi^{-1}(i)$ is prescribed in advance.

In a Tverberg-type problem one looks for a collection $\{S_i\}_{i=1}^r$ of r disjoint subsets of S such that the intersection of all convex sets $\mathrm{conv}(S_i)$ is non-empty. Such a collection can be recorded (visualized) as a single set (simplex) $X \subset [m]\times[r]$ in the 'chessboard' $[m] \times [r]$ where by definition $X = \{(i, j) \mid x_i \in S_j\}$. The set X may satisfy some additional condition expressing its compatibility with the structure $\mathcal{S}$. For example if S is colored by a coloring function ϕ then the usual condition is that S_i is a 'rainbow set' in the sense that it has at most one point in each color, $|S_i \cap C_j| \leq 1$ for each i and j. More generally, for a chosen simplicial complex $K \subset 2^{[m]}$, the structure $\mathcal{S}$ may arise from a simplicial map $f : K \to \mathbb{R}^d$

where $S = f(\text{Vert}(K))$ and the compatibility condition says that $S_i \in K$ for each $i = 1, \dots, r$.

The ***chessboard transform*** of an $\mathcal{S}$-structured set S is the simplicial complex $\Delta_{m,q}[\mathcal{S}]$ which collects all $\mathcal{S}$-constrained ($\mathcal{S}$-compatible) subsets $X \subset [m] \times [r]$. If S is not structured, i.e. if there are no other constraints on X aside from the requirement that $S_i \cap S_j = \emptyset$ for $i \neq j$, we recover the complex $\mathcal{N}_{m,r}$ from Example 22.1.4. If $\mathcal{S}$ is the structure associated to a coloring function ϕ then the complex $\Delta_{m\times q}[\mathcal{S}]$ is the join,

$$\Delta_{m,q}[\mathcal{S}] = \Delta_{m_1,q} * \dots * \Delta_{m_k,q}$$

of standard chessboard complexes [BLVŽ94, VŽ11]. By definition the (standard) chessboard complex $\Delta_{p,q}$ is the complex of all non-attacking placements of rooks in a $(p \times q)$-chessboard (a placement is non-attacking (non-taking) if it is not allowed to have more than one rook in the same row or in the same column).

22.2 PARTITIONS OF MASS DISTRIBUTIONS

The problem of consensus division arises when two or more competitive or cooperative parties, each guided by their own preferences, divide an object according to some notion of fairness. There are many different mathematical reformulations of this problem depending on what kind of divisions are allowed, what kind of object is divided, whether the parties involved are cooperative or not, etc. Early examples of problems and results of this type are the "ham sandwich theorem" of Steinhaus and Banach, the "envy-free cake division problem" of Selfridge and Conway, and the "proportional cake-division problem" of Steinhaus, see [BT95]. The problems of partitioning mass distributions in Euclidean spaces, as a mathematical reformulation of the problem of fair division, form the first circle of discrete geometric problems where topological methods have been applied with great success.

GLOSSARY

Measure: A non-negative, countably additive (σ-additive) Borel measure, defined on the σ-algebra generated by all open (closed) sets in $\mathbb{R}^d$. A measure is finite if $\mu(\mathbb{R}^d) < +\infty$.

Measurable set: Any set in the domain of the function μ.

Mass distribution and density function: A density function is an integrable function $f : \mathbb{R}^d \to [0, +\infty)$ representing the density of a "mass distribution" (measure) on $\mathbb{R}^d$. The measure μ arising this way is defined by $\mu(A) := \int_A f\, dx$.

Counting measure: If $S \subset \mathbb{R}^d$ is a finite set, then $\mu(A) := |A \cap S|$ is the *counting measure* induced by the set S.

Halving hyperplane: A hyperplane that simultaneously bisects a family of measurable sets.

22.2.1 THE HAM SANDWICH THEOREM

Given a collection of d measurable sets (mass distributions, finite sets) in $\mathbb{R}^d$, the problem is to simultaneously bisect all of them by a single hyperplane.

THEOREM 22.2.1 *Ham Sandwich Theorem* [Bor33]

Let $\mu_1, \mu_2, \ldots, \mu_d$ be a collection of measures (mass distributions, measurable sets, finite sets). Then there exists a hyperplane H such that for all $i = 1, \ldots, d$, $\mu_i(H^+) \geq 1/2\, \mu_i(\mathbb{R}^d)$ and $\mu_i(H^-) \geq 1/2\, \mu_i(\mathbb{R}^d)$, where H^+ and H^- are the closed halfspaces associated with the hyperplane H.

In the special case where $\mu(H) = 0$, i.e., when the hyperplane itself has measure zero, H is called a *halving hyperplane* since $\mu_i(H^+) = \mu_i(H^-) = 1/2\, \mu_i(\mathbb{R}^d)$ for all i. A halving hyperplane H is also called a "ham sandwich cut", in agreement with the popular interpretation of the 3-dimensional case where a 'sandwich' is cut into two equal parts by a straight cut of a knife.

TOPOLOGICAL BACKGROUND

The topological result lying behind the ham sandwich theorem is the Borsuk-Ulam theorem, [Ste85, Mat08]. The proof of the ham sandwich theorem historically marks one of the first applications of the CS/TM-scheme, with the $(d-1)$-sphere as the configuration space, $\mathbb{R}^d$ as the test space, and $G = \mathbb{Z}_2$ as the group of symmetries associated to the problem. Given a collection $\{A_i\}_{i=1}^d$ of d measurable sets, the test map $t : S^{d-1} \to \mathbb{R}^d$ is defined by $t(e) = (\alpha_1, \ldots, \alpha_d)$, with α_i determined by the condition that $H_i := \{x \in \mathbb{R}^d \mid \langle x, e\rangle = \alpha_i\}$ is a median halving hyperplane for the measurable set A_i. (The median halving hyperplane in any direction is the mid-hyperplane between the two extreme halving hyperplanes in that direction.) The test space is the diagonal $Z := \{(\alpha, \ldots, \alpha) \in \mathbb{R}^d \mid \alpha \in \mathbb{R}\}$. The test map t is obviously "odd", or $\mathbb{Z}_2$-equivariant, in the sense that $t(-e) = -t(e)$. The Borsuk-Ulam theorem is applied to the modified test map $\bar{t} = p \circ t : S^{n-1} \to Z^\perp$, where $Z^\perp \cong \mathbb{R}^{n-1}$ is the hyperplane orthogonal to Z and $p : \mathbb{R}^n \to Z^\perp$ is the associated orthogonal projection.

THEOREM 22.2.2 *Borsuk-Ulam Theorem* [Bor33]

For every continuous map $f : S^n \to \mathbb{R}^n$ from an n-dimensional sphere into n-dimensional Euclidean space, there exists a point $x \in S^n$ such that $f(x) = f(-x)$.

There is a different topological approach to the ham sandwich theorem closer to the earlier example about a watch with two indistinguishable hands. Here we mention only that the role of the torus T^2 is played by a manifold M representing all hyperplanes in $\mathbb{R}^d$ (the configuration space), while the curves Γ_1 and Γ_2 are replaced by suitable submanifolds N_i of M, one for each of the measures μ_i, $i = 1, \ldots, d$. N_i is defined as the space of all halving hyperplanes for the measurable set A_i.

22.2.2 THE CENTER POINT THEOREM

THEOREM 22.2.3 *Center Point Theorem* [Rad46]

Let μ be one of the measures described in Section 22.2.1. Then there exists a point $x \in \mathbb{R}^d$ such that for every closed halfspace $P \subset \mathbb{R}^d$, if $x \in P$ then

$$\mu(P) \geq \frac{\mu(\mathbb{R}^d)}{d+1}.$$

TOPOLOGICAL BACKGROUND

If the Borsuk-Ulam theorem is responsible for the ham sandwich theorem, then R. Rado's center point theorem can be seen as a consequence of another well-known topological result, *Brouwer's fixed point theorem* [Bro12].

In a nutshell, the center point theorem is, up to some technical details, deduced from Brouwer's theorem as follows. If x is not a center point of μ then there is a halfspace $P \ni x$ such that $\mu(P) < \mu(\mathbb{R}^d)/(d+1)$. The map $x \mapsto x + e$, where $e \in S^{d-1}$ is the outer normal unit vector of P, defines a vector field without zeros (contradicting the Brouwer's theorem).

Recall that the center point theorem was originally deduced (by R. Rado) from Helly's theorem about intersecting families of convex sets, which also has several topological relatives. For this reason, it is often viewed as a measure-theoretic equivalent of Helly's theorem.

APPLICATIONS AND RELATED RESULTS

As noted by Miller and Thurston (see [MTTV97, MTTV98]), the center point theorem and the Koebe theorem on the disk representation of planar graphs can be used to prove the existence of a small separator for a planar graph, a result proved originally (by Lipton and Tarjan) by different methods.

The ***regression depth*** $\mathrm{rd}_{\mathcal{P}}(H)$ of a hyperplane H relative to a collection $\mathcal{P}$ of n points in $\mathbb{R}^d$ is the minimum number of points that H must pass through in moving to the vertical position. Dually, given an arrangement $\mathcal{H}$ of n hyperplanes in $\mathbb{R}^d$, the regression depth $\mathrm{rd}_{\mathcal{H}}(x)$ of a point x relative to $\mathcal{H}$ is the smallest k such that x cannot escape to infinity without crossing (or moving parallel to) at least k hyperplanes. The problem of finding a point (resp. hyperplane) with maximum regression depth relative to $\mathcal{H}$ (resp. $\mathcal{P}$) is shown in [AET00] to be intimately connected with the problem of finding center points. The main result (confirming a conjecture of Rousseeuw and Hubert) is that there always exists a point with regression depth $\lceil n/(d+1)\rceil$; cf. Chapter 60 of this Handbook.

The 'dual center point' theorem [Kar09] says that for each general position family of n hyperplanes in $\mathbb{R}^d$ there exists a point c such that any ray starting at c intersects at least $\lceil n/d+1\rceil$ hyperplanes. Both this result and the center point theorem are special cases of the 'projective center point theorem' proved by Karasev and Matschke (see [KM14, Theorems 1.2 and 1.3]).

22.2.3 CENTER TRANSVERSAL THEOREM

THEOREM 22.2.4 *Center Transversal Theorem* [ŽV90]

Let $0 \le k \le d-1$ and suppose that $\mu_0, \mu_1, \ldots, \mu_k$ is a collection of finite, Borel measures on $\mathbb{R}^d$. Then there exists a k-dimensional affine subspace $D \subset \mathbb{R}^d$ such that for every closed halfspace $H(v,\alpha) := \{x \in \mathbb{R}^d \mid \langle x, v\rangle \le \alpha\}$ and every $i \in \{0,1,\ldots,k\}$,

$$D \subset H(v,\alpha) \Longrightarrow \mu_i(H(v,\alpha)) \ge \frac{\mu_i(\mathbb{R}^d)}{d-k+1}.$$

TOPOLOGICAL BACKGROUND

The center transversal theorem contains the ham sandwich and center point theorems as two boundary cases [ŽV90]. The topological principle that is at the root of this result should be strong enough for this purpose. This result has several incarnations. One of them is a theorem of E. Fadell and S. Husseini [FH88] that claims the nonexistence of a $\mathbb{Z}_2^{\oplus k}$-equivariant map $f : V_{n,k} \to (\mathbb{R}^k)^{n-k} \setminus \{0\}$ from the Stiefel manifold of all orthonormal k-frames in $\mathbb{R}^n$ to the sum of $n-k$ copies of $\mathbb{R}^k$. The group $\mathbb{Z}_2^{\oplus k}$ can be identified with the group of all diagonal, orthogonal matrices and its action on $\mathbb{R}^k$ is induced by the obvious action of $O(k)$. A related result [FH88, ŽV90] is that the vector bundle $\xi_k^{\oplus(n-k)}$ does not admit a nonzero, continuous cross-section, where ξ_k is the tautological k-plane bundle over the Grassmann manifold $G_k(\mathbb{R}^n)$. For the introduction to vector bundles, classical Lie groups and associated manifolds the reader is referred to [Bre93, Chapters II and VII].

APPLICATIONS AND RELATED RESULTS

The following Helly-type transversal theorem, due to Dol'nikov, is a consequence of the same topological principle that is at the root of the center transversal theorem. Moreover, the center transversal theorem is related to Dol'nikov's result in the same way that the center point theorem is related to Helly's theorem.

THEOREM 22.2.5 [Dol93]

Let $\mathcal{K}_0, \dots, \mathcal{K}_k$ be families of compact convex sets. Suppose that for every i, and for each k-dimensional subspace $V \subset \mathbb{R}^d$, there exists a translate V_i of V intersecting every set in $\mathcal{K}_i$. Then there exists a common k-dimensional transversal of the family $\mathcal{K} := \bigcup_{i=0}^{k} \mathcal{K}_i$, i.e., there exists an affine k-dimensional subspace of $\mathbb{R}^d$ intersecting all the sets in $\mathcal{K}$.

Let $\mathcal{K} = \{K_0, \dots, K_k\}$ be a family of convex bodies in $\mathbb{R}^n$, $1 \le k \le n-1$. Then an affine l-plane $A \subset \mathbb{R}^n$ is called a ***common maximal l-transversal*** of $\mathcal{K}$ if $m(K_i \cap A) \ge m(K_i \cap (A+x))$ for each $i \in \{0, \dots, k\}$ and each $x \in \mathbb{R}^n$, where m is l-dimensional Lebesgue measure in A and $A+x$, respectively. It was shown in [MVŽ01] that, given a family $\mathcal{K} = \{K_i\}_{i=0}^{k}$ of convex bodies in R^n $(k < l)$, the set $C_l(\mathcal{K})$ of all common maximal l-transversals of $\mathcal{K}$ has to be "large" from both the measure-theoretic and the topological point of view.

22.2.4 EQUIPARTITION OF MASSES BY HYPERPLANES

Every measurable set $A \subset \mathbb{R}^3$ can be partitioned by three planes into 8 pieces of equal measure (H. Hadwiger). This is an instance of the general problem of characterizing all triples (d, j, k) such that for any j mass distributions (measurable sets) in $\mathbb{R}^d$, there exist k hyperplanes such that each of the 2^k 'orthants' contains the fraction $1/2^k$ of each of the masses. Such a triple (d, j, k) will be called *admissible*. Let $\Delta(j, k)$ be the minimum dimension d such that the triple (d, j, k) is admissible. It is known (E. Ramos, [Ram96]) that $d \ge j(2^k - 1)/k$ is a necessary condition and $d \ge j2^{k-1}$ a sufficient one for a triple (d, j, k) to be admissible. Ramos's method yields many interesting results in lower dimensions, including the admissibility of the triples $(9,3,3)$, $(9,5,2)$, and $(5,1,4)$, see [Živ15, BFHZ15, BFHZ15, VŽ15] for subsequent developments. The most interesting special case that is still out of reach is the triple $(4,1,4)$.

PROBLEM 22.2.6

Is it true that each measurable set $A \subset \mathbb{R}^4$ can be partitioned by four hyperplanes into sixteen parts of equal measure?

THEOREM 22.2.7 [Živ15] (see also [BFHZ15])

Each collection of $j = 4 \cdot 2^\nu + 1$ measures in $\mathbb{R}^d$ where $d = 6 \cdot 2^\nu + 2$ admits an equipartition by two hyperplanes. From here we deduce that,

$$\Delta(4 \cdot 2^\nu + 1, 2) = 6 \cdot 2^\nu + 2 \tag{22.2.1}$$

since following [Ram96] *the lower bound $\Delta(j,2) \geq 3j/2$ holds for each $j \geq 1$.*

APPLICATIONS AND RELATED RESULTS

According to [Mat08], an early interest of computer scientists in partitioning mass distributions by hyperplanes was stimulated in part by *geometric range searching*; cf. Chapter 41 of this Handbook. More general equipartitions involve not necessarily central hyperplane arrangements. An example is a result of S. Vrećica [Vre09, Theorem 3.1] which says that a mass distribution in $\mathbb{R}^d$ admits an equipartition by $4d-2$ polyhedral regions determined by a collection of $2d-2$ parallel hyperplanes and an additional transverse hyperplane. The existence of very general equipartitions made by iterated hypeplane cuts, where the directions of the hyperplanes were prescribed in advance, was established in [KRPS16]. This result can be also classified as a relative of the 'splitting necklace theorem' (Theorem 22.4.3) since the equipartition is obtained after assembling the polyhedral regions into k collections (k is the number of parties involved in the consensus division).

22.2.5 PARTITIONS BY CONICAL, POLYHEDRAL FANS

An old result of R. Buck and E. Buck (*Math. Mag.*, 1949) says that for each continuous mass distribution in the plane, there exist three concurrent lines $l_1, l_2, l_3 \subset \mathbb{R}^2$ that partition $\mathbb{R}^2$ into six sectors of equal measure. It is natural to search for higher dimensional analogs of this result.

Suppose that $Q \subset \mathbb{R}^d$ is a convex polytope and assume that the origin $O \in \mathbb{R}^d$ belongs to the interior $\mathrm{int}(Q)$ of Q. Let $\{F_i\}_{i=1}^k$ be the collection of all facets of Q. Let $\mathcal{F} := \mathrm{fan}(Q)$ be the associated ***fan***, i.e., $\mathcal{F} = \{C_1, \dots, C_k\}$ where $C_i = \mathrm{cone}(F_i)$ is the convex closed cone with vertex O generated by F_i.

THEOREM 22.2.8 [Mak01]

Let Q be a regular dodecahedron with the origin $O \in \mathbb{R}^3$ as its barycenter. Then for any continuous mass distribution μ on $\mathbb{R}^3$, centrally symmetric with respect to O, there exists a linear map $L \in GL(3,\mathbb{R})$ such that

$$\mu(L(C_1)) = \mu(L(C_2)) = \ldots = \mu(L(C_{12})).$$

Makeev actually showed in [Mak01] that L can be found in the set of all matrices of the form $a \cdot t$, where t is an upper triangular matrix and $a \in GL(3, \mathbb{R})$ is a matrix given in advance. In an earlier paper (see [Mak98]) he showed that a radial partition by a fan determined by the facets of a cube always exists for an arbitrary measure in $\mathbb{R}^3$. Moreover, he shows in [Mak01] that a result analogous to Theorem 22.2.8 also holds for rhombic dodecahedra. Recall that the rhombic dodecahedron U_3 is the polytope bounded by twelve planes, each containing an edge of a cube and parallel to one of the great diagonal planes. A higher dimensional analogue of the rhombic dodecahedron is the polytope U_n in $\mathbb{R}^n$ described as the dual of the difference body of a regular simplex.

PROBLEM 22.2.9

Let $T \subset \mathbb{R}^n$ be a regular simplex and $Q := T - T$ the associated "difference polytope." Let $U_n := Q^\circ$ be the polytope polar to Q. Clearly U_n is a centrally symmetric polytope with $n^2 + n$ facets F_i, $i = 1, \dots, n^2 + n$. Let $\{K_i\}_{i=1}^{n^2+n}$ be the associated conical dissection of $\mathbb{R}^n$, where $K_i := \mathrm{cone}(F_i)$. Is it true that for any continuous mass distribution μ on $\mathbb{R}^n$ there exists a nondegenerate affine map $A : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\mu(A(K_1)) = \mu(A(K_2)) = \ldots = \mu(A(K_{n^2+n}))\,?$$

The center transversal theorem is a common generalization of the ham sandwich theorem and the center point theorem. There are other general statements of this type as illustrated by the following conjecture. For a given non-degenerate simplex $\Delta := \mathrm{conv}(\{a_i\}_{i=0}^m) \subset \mathbb{R}^d$ let $\mathcal{D}(\Delta) = \{D_i\}_{i=0}^m$ be the associated dissection of $\mathbb{R}^d$ into $m+1$ wedgelike cones, where $D_i := P^\perp \oplus \mathrm{cone}(\mathrm{conv}(\{a_j\}_{j \neq i}))$ and $P := \mathrm{aff}\, \Delta$.

CONJECTURE 22.2.10 [VŽ92]

Let $\mu_0, \dots, \mu_k$ be a family of continuous mass distributions (measures), $0 \leq k \leq d-1$, defined on $\mathbb{R}^d$. Then there exists a $(d-k)$-dimensional regular simplex Δ such that for the corresponding dissection, $\mathcal{D}(\Delta)$, for some $x \in \mathbb{R}^d$, and for all i, j,

$$\mu_i(x + D_j) \geq \frac{\mu_i(\mathbb{R}^d)}{d-k+1}.$$

Both $B(d, 0)$ and $B(d, d-1)$ are true ($B(d, d-1)$ is the ham sandwich theorem). The conjecture is also confirmed [VŽ92] in the case $B(d, d-2)$ for all d. Moreover, there exists a natural topological conjecture implying $B(d, k)$ that is closely related to the analogous statement needed for the center transversal theorem. This statement essentially claims that there is no $\mathbb{Z}_{k+1}$-equivariant map from the Stiefel manifold $V_k(\mathbb{R}^n)$ to the unit sphere $S(V)$ in an appropriate $\mathbb{Z}_{k+1}$-representation V.

22.2.6 PARTITIONS BY CONVEX SETS

THEOREM 22.2.11 [Sob12] [KHA14]

Let n and d be integers with $n, d \geq 2$. Assume that $\mu_1, \dots, \mu_d$ are continuous mass distributions such that $\mu_1(\mathbb{R}^d) = \ldots = \mu_d(\mathbb{R}^d) = n$. Then there exists a partition of $\mathbb{R}^d$ into n sets $C_1, \dots, C_n$ such that the interiors $\mathrm{int}(C_i)$ are convex sets and $\mu_i(C_j) = 1$ for each $i = 1, \dots, n$ and $j = 1, \dots, n$.

The planar version of Theorem 22.2.11 was conjectured by Kaneko and Kano [KK99] and proved by a number of authors (see [BM01]) before it was established in full generality by P. Soberón [Sob12] and R. Karasev, A. Hubard, and B. Aronov [KHA14].

One of the key steps in the proof of Theorem 22.2.11 was the recognition of *weighted Voronoi diagrams* as elements of the configuration space adequate for this problem (Example 22.1.7). It was quickly recognized that the method has a potential for proving other results of similar nature (with equal or similar configuration space). Note however that some of these results are known to hold only under the assumption that n (the number of convex sets in the partition) is a prime power.

THEOREM 22.2.12 [KHA14, Corollary 1.1] (see also [BZ14, Zie15])

Given a convex body K in $\mathbb{R}^d$ and a prime power $n = p^k$ it is possible to partition K into n convex bodies with equal d-dimensional volumes and equal $(d-1)$-dimensional surface areas.

TOPOLOGICAL BACKGROUND

The ultimate topological statement responsible for all known results about convex equipartitions is the following far reaching theorem about maps equivariant with respect to the symmetric group Σ_n. Special cases of the result were originally proved by D. Fuchs (the case $d = 2, p = 2$) and V. Vassiliev (the case $d = 2$ and $p \geq 3$). The result was in full generality (following the general ideas of the proofs of Fuchs and Vassiliev) established by R. Karasev (see [KHA14]). A proof based on different ideas can be found in [BZ14], see also [Zie15] for an outline and some additional information.

THEOREM 22.2.13 (D. Fuchs, V. Vassiliev, R. Karasev) (see also [BZ14])

Let $F_n(\mathbb{R}^d)$ be the classical configuration space of all ordered n-tuples of distinct points in $\mathbb{R}^d$ where $n = p^k$ is a prime power. Suppose that $\mathrm{Mat}_0(d-1, n) \cong \mathbb{R}^{(d-1)(n-1)}$ is the vector space of all $(d-1) \times n$ matrices such that the entries in each of the $(d-1)$ rows add up to zero. Suppose that the symmetric group Σ_n acts by permuting the coordinates of $\overline{x} \in F_n(\mathbb{R}^d)$ and the columns of the matrix $m = (m_{ij}) \in \mathrm{Mat}_0(d-1, n)$. Then for each Σ_n-equivariant map $f : F_n(\mathbb{R}^d) \to \mathrm{Mat}_0(n, d-1)$ there exists a configuration $\overline{x}$ such that $f(\overline{x}) = 0$.

APPLICATIONS AND RELATED RESULTS

It is known from (first order) equivariant obstruction theory that the (non)existence of a Σ_n-equivariant map is closely related to the question of (non)existence of H-equivariant maps for different Sylow subgroups of the symmetric group Σ_n. If $n = 2^k$ then the 2-Sylow subgroup of Σ_n is precisely the group of all automorphisms of the binary tree described in Example 22.1.6. This observation establishes a link between Theorem 22.2.13 and Gromov's 'Non-vanishing Lemma', [Gro03, p. 188].

22.2.7 PARTITIONS BY CONVEX SETS IN PRESCRIBED RATIOS

The conjecture of Kaneko and Kano (Section 2.2.6) motivated I. Bárány and J. Matoušek [BM01, BM02] to study general conical partitions of planar or spherical measures in prescribed ratios. Here we assume that all measures are continuous mass distributions.

An arrangement of k semilines in the Euclidean plane or on the 2-sphere is called a k-fan if all semilines start from the same point. A k-fan is an α-partition for a probability measure μ if $\mu(\sigma_i) = \alpha_i$ for each $i = 1, \dots, k$, where $\{\sigma_i\}_{i=1}^k$ are conical sectors associated with the k-fan and $\alpha = (\alpha_1, \dots, \alpha_k)$ is a given vector. The set of all $\alpha = (\alpha_1, \dots, \alpha_m)$ such that for any collection of probability measures $\mu_1, \dots, \mu_m$ there exists a common α-partition by a k-fan is denoted by $\mathcal{A}_{m,k}$. It was shown in [BM01] that the interesting cases of the problem of existence of α-partitions are $(k, m) = (2, 3), (3, 2), (4, 2)$.

CONJECTURE 22.2.14 [BM01, BM02]

Suppose that (k, m) is equal to $(2, 3), (3, 2)$ or $(4, 2)\}$. Then $\alpha \in \mathcal{A}_{k,m}$ if and only if

$$\alpha_1 + \ldots + \alpha_m = 1 \quad \text{and} \quad \alpha_i > 0 \quad \text{for each} \quad i = 1, \ldots m.$$

Bárány and Matoušek proposed a very nice approach to this problem [BM01, BM02] and showed that $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})$ are in $\mathcal{A}_{4,2}$. From here they deduced that $\{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})\} \cup \{(\frac{p}{5}, \frac{q}{5}, \frac{r}{5}) \mid p, q, r \in N^+, p + q + r = 5\} \subset \mathcal{A}_{3,2}$.

22.3 THE PROBLEMS OF BORSUK AND KNASTER

The topological methods used in proofs of measure partition results are actually applicable to a much wider class of combinatorial and geometric problems. In fact quite different problems, which on the surface have very little in common may actually lead to the same or closely related configuration spaces and test maps. This in turn implies that such problems both follow from the same general topological principle and that they could, despite appearances, be classified as "relatives".

22.3.1 BORSUK'S PROBLEM

Borsuk's well-known problem [Bor33] about covering sets in $\mathbb{R}^n$ with sets of smaller diameter was solved in the negative by J. Kahn and G. Kalai [KK93] who proved that the size of a minimal cover is exponential in n; see Chapters 1 and 2 of this Handbook. This, however, gave a new impetus to the study of "Borsuk numbers" after the old exponential upper bounds suddenly became more plausible. This may be one of the reasons why results about "universal covers", originally used for these estimates, have received new attention.

The following result was proved originally by V. Makeev [Mak98]; see also [HMS02, Kup99]. Recall the rhombic dodecahedron U_3, the polytope bounded by twelve rhombic facets, which appeared in Section 22.2.5.

THEOREM 22.3.1 [Mak98] (see also [HMS02, Kup99])

A rhombic dodecahedron of width 1 is a universal cover for all sets $S \subset \mathbb{R}^3$ of diameter 1. In other words, each set of diameter 1 in 3-space can be covered by a rhombic dodecahedron whose opposite faces are 1 unit apart.

Let $\Sigma \subset \mathbb{R}^n$ be a regular simplex of edge-length 1, with vertices $v_1, \ldots, v_{n+1}$. Then the intersection of $n(n+1)/2$ parallel strips S_{ij} of width 1, where S_{ij} is bounded by the $(n-1)$-planes orthogonal to the segment $[v_i, v_j]$ passing through the vertices v_i and v_j $(i < j)$, is a higher dimensional analog of the rhombic dodecahedron. It is easy to see that this is just another description of the polytope U_n that we encountered in Problem 22.2.9.

CONJECTURE 22.3.2 *Makeev's conjecture* [Mak94]

The polytope U_n is a universal cover *in $\mathbb{R}^n$. In other words, for each set $S \subset \mathbb{R}^n$ of diameter 1, there exists an isometry $I : \mathbb{R}^n \to \mathbb{R}^n$ such that $S \subset I(U_n)$.*

The relevance of Makeev's conjecture for the general Borsuk problem is obvious since in low dimensions, $d = 2$ and $d = 3$, the solutions were based on the construction of suitable universal covers. (Note that the case $d = 4$ of the Borsuk partition problem is still open!) The following stronger conjecture is yet another example of a topological statement with potentially interesting consequences in discrete and computational geometry.

CONJECTURE 22.3.3 [HMS02]

Let $f : S^{n-1} \to \mathbb{R}$ be an odd function, and let $\Delta^n \subset \mathbb{R}^n$ be a regular simplex of edge-length 1, with vertices $v_1, \ldots, v_{n+1}$. Then there exists an orthogonal linear map $A \in SO(n)$ such that the $n(n+1)/2$ hyperplanes H_{ij}, $1 \leq i < j \leq n+1$, are concurrent, where

$$H_{ij} := \{x \in \mathbb{R}^n \mid \langle x, A(v_j - v_i)\rangle = f(A(v_j - v_i))\}.$$

G. Kuperberg showed in [Kup99] that, unlike the cases $n = 2$ and $n = 3$, for $n \geq 4$ there is homologically an even number of isometries $I : \mathbb{R}^n \to \mathbb{R}^n$ such that $S \subset I(U_n)$ for a given set S of constant width. Kuperberg showed that the Makeev conjecture can be reduced (essentially in the spirit of the CS/TM-scheme) to the question of the existence of a Γ-equivariant map $f : SO(n) \to V \setminus \{0\}$, where Γ is a group of symmetries of the root system of type A_n and the test space V is an $n(n-1)/2$-dimensional representation of Γ. The fact that such a map exists if and only if $n \geq 4$ may be an indication that the Makeev conjecture is false in higher dimensions.

22.3.2 KNASTER'S PROBLEM

Knaster's problem (Problem 4., *Colloq. Math.*, 1:30, 1947) is one of the old conjectures of discrete geometry with a distinct topological flavor. The conjecture is now known to be false in general, but the problem remains open in many interesting special cases.

PROBLEM 22.3.4 *Knaster's problem*

Given a finite subset $S = \{s_1, \ldots, s_k\} \subset S^n$ of the n-sphere, determine the conditions on k and n so that for each continuous map $f : S^n \to \mathbb{R}^m$ there will exist an isometry $O \in SO(n+1)$ with

$$f(O(s_1)) = f(O(s_2)) = \ldots = f(O(s_k)).$$

B. Knaster originally conjectured that such an isometry O always exists if $k \leq n-m+2$. Just as in the case of the Borsuk problem, the first counterexamples took a long time to appear. V. Makeev [Mak86, Mak90], and somewhat later K. Babenko and S. Bogatyi [BB89], showed that the condition $k \leq n-m+2$ is not sufficient, at least for some special values of n and m. New examples were discovered by W. Chen (*Topology*, 1998), B.S. Kashin and S.J. Szarek (*C. R. Math. Acad. Sci. Paris*, 2003), A. Hinrichs and C. Richter (*Israel J. Math*, 2005), see also [BK14] for an update and related results.

The fact that Knaster's conjecture is false in general does not rule out the possibility that for some special configurations $S \subset S^n$ the answer is still positive. The case where S is the set of vertices of a largest regular simplex inscribed in S^n is of special interest since it directly generalizes the Borsuk-Ulam theorem.

Questions closely related to Knaster's conjecture are the problems of inscribing or circumscribing polyhedra to convex bodies in $\mathbb{R}^n$; see [HMS02, Kup99]. G. Kuperberg observed that both the circumscription problem for constant-width bodies and Knaster's problem are special cases of the following problem.

PROBLEM 22.3.5 [Kup99]

Given a finite set T of points on S^{d-1} and a linear subspace L of the space of all functions from T to $\mathbb{R}^n$, decide if, for each continuous function $f : S^{d-1} \to \mathbb{R}^n$, there is an isometry O such that the restriction of $f \circ O$ to T is an element of L.

22.4 TVERBERG-TYPE THEOREMS

GLOSSARY

Tverberg-type problem: A problem in which a finite set $A \subset \mathbb{R}^d$ is to be partitioned into nonempty, disjoint pieces $A_1, \ldots, A_r$, possibly subject to some constraints, so that the corresponding convex hulls $\{\mathrm{conv}(A_i)\}_{i=1}^r$ intersect.

Colors: A set of $k+1$ colors is a collection $\mathcal{C} = \{C_0, \ldots, C_k\}$ of disjoint subsets of $\mathbb{R}^d$. A set $B \subset \mathbb{R}^d$ is ***properly colored*** if it contains at most one point from each of the sets C_i; in this case $\mathrm{conv}(B)$ is called a ***rainbow simplex*** (possibly degenerate).

Types A, B, and ***C***: Colored Tverberg problems are of type A, type B, or type C depending on whether $k = d, k < d$ or $k > d$ (resp.), where $k+1$ is the number of colors and d the dimension of the ambient space.

Tverberg numbers $T(r,d)$, $T(r,k,d)$: $T(r,k,d)$ is the minimal size of each of the colors C_i, $i = 0, \ldots, k$, that guarantees that there always exist r intersecting, pairwise vertex-disjoint, rainbow simplices. $T(r,d) := T(r,d,d)$.

AN OVERVIEW AND FIRST EXAMPLES

'Tverberg problems' is a common name for a class of theorems and conjectures about finite sets of points (point clouds) in $\mathbb{R}^d$. The original Tverberg theorem (Theorem 22.4.1) claims that every set $K \subset \mathbb{R}^d$ with $(r-1)(d+1)+1$ elements can be partitioned $K = K_1 \cup \ldots \cup K_r$ into r nonempty, pairwise disjoint subsets $K_1, \ldots, K_r$ such that the corresponding convex hulls have a nonempty intersection:

$$\bigcap_{i=1}^{r} \mathrm{conv}(K_i) \neq \emptyset. \tag{22.4.1}$$

This result can be reformulated as the statement that for each linear (affine) map $f : \Delta^D \xrightarrow{a} \mathbb{R}^d$ $(D = (r-1)(d+1))$ there exist r nonempty disjoint faces $\Delta_1, \ldots, \Delta_r$ of the simplex Δ^D such that $f(\Delta_1) \cap \ldots \cap f(\Delta_r) \neq \emptyset$. This form of Tverberg's result can be abbreviated as follows,

$$(\Delta^{(r-1)(d+1)} \xrightarrow{a} \mathbb{R}^d) \Rightarrow (r - \text{intersection}). \tag{22.4.2}$$

Here we tacitly assume that the faces intersecting in the image are always vertex disjoint. The letter "a" over the arrow means that the map is affine and its absence indicates that it can be an arbitrary continuous map.

It is desirable to refine the original Tverberg theorem by specifying which simplicial complexes K can replace the full simplex $\Delta^{(r-1)(d+1)}$ in the proposition (22.4.2).

The following four statements are illustrative for results of 'colored Tverberg type'.

$$(K_{3,3} \longrightarrow \mathbb{R}^2) \Rightarrow (2 - \text{intersection}) \tag{22.4.3}$$

$$(K_{3,3,3} \xrightarrow{a} \mathbb{R}^2) \Rightarrow (3 - \text{intersection}) \tag{22.4.4}$$

$$(K_{5,5,5} \longrightarrow \mathbb{R}^3) \Rightarrow (3 - \text{intersection}) \tag{22.4.5}$$

$$(K_{4,4,4,4} \longrightarrow \mathbb{R}^3) \Rightarrow (4 - \text{intersection}) \tag{22.4.6}$$

$K_{t_1,t_2,\ldots,t_k} = [t_1] * [t_2] * \ldots * [t_k]$ is by definition the complete multipartite simplicial complex obtained as a join of 0-dimensional complexes (finite sets). By definition the vertices of this complex are naturally partitioned into groups of the same 'color'. For example $K_{p,q} = [p] * [q]$ is the complete bipartite graph obtained by connecting each of p 'red vertices' with each of q 'blue vertices'. The simplices of $K_{t_1,t_2,\ldots,t_k}$ are often referred to as *rainbow simplices.*

The implication (22.4.3) says that for each continuous map $\phi : K_{3,3} \to \mathbb{R}^2$ there always exist two vertex-disjoint edges which intersect in the image. In light of the Hanani-Tutte theorem this statement is equivalent to the non-planarity of the complete bipartite graph $K_{3,3}$. The implication (22.4.4) is an instance of a result of Bárány and Larman [BL92]. It says that each collection of nine points in the plane, evenly colored by three colors, can be partitioned into three multicolored or 'rainbow triangles' which have a common point. Note that a 9-element set $C \subset \mathbb{R}^2$ which is evenly colored by three colors, can be also described by a map $\alpha : [3] \sqcup [3] \sqcup [3] \to \mathbb{R}^2$ from a disjoint sum of three copies of [3]. In the same spirit an affine map $\phi : K_{3,3,3} \xrightarrow{a} \mathbb{R}^2$ parameterizes not only the colored set itself but takes into account from the beginning that some simplices (multicolored or rainbow simplices) play a special role.

A similar conclusion has statement (22.4.5) which is a formal analogue of the statement (22.4.3) in dimension 3. It is an instance of a result of Vrećica and Živaljević [VŽ94], which claims the existence of three intersecting, vertex-disjoint rainbow triangles in each constellation of 5 red, 5 blue, and 5 white stars in the 3-space.

A non-linear version of this result is that $K_{5,5,5}$ is 3-non-embeddable in $\mathbb{R}^3$ in the sense that there always exists a triple point in the image.

The statement (22.4.6) is an instance of the result of Blagojević, Matschke, and Ziegler [BMZ15] saying that 4 intersecting, vertex disjoint rainbow tetrahedra in $\mathbb{R}^3$ will always appear if we are given sixteen points, evenly colored by four colors.

All statements (22.4.3)–(22.4.6) are instances of results of colored Tverberg type. They are respectively classified as the Type A ((22.4.4) and (22.4.6)), and the Type B ((22.4.3) and (22.4.5)) colored Tverberg results.

22.4.1 MONOCHROMATIC TVERBERG THEOREMS

THEOREM 22.4.1 *Affine Tverberg Theorem* [Tve66]

Every set $K = \{a_j\}_{j=0}^{(r-1)(d+1)} \subset \mathbb{R}^d$ *with* $(r-1)(d+1)+1$ *elements can be partitioned into* r *nonempty, disjoint subsets* $K_1, \ldots, K_r$ *so that the corresponding convex hulls have nonempty intersection:*

$$\bigcap_{i=1}^{r} \operatorname{conv}(K_i) \neq \emptyset\,.$$

(The special case $q = 2$ is Radon's theorem; see Chapter 4.)

THEOREM 22.4.2 *Topological Tverberg Theorem* [BSS81, Öza87]

Assume that r *is a prime* ([BSS81]) *or a prime power* ([Öza87])*. Then for every continuous map* $f : \Delta^{(r-1)(d+1)} \to \mathbb{R}^d$ *there exist vertex-disjoint faces* $\Delta_1, \ldots, \Delta_r \subset \Delta^{(r-1)(d+1)}$ *such that* $\bigcap_{i=1}^{r} f(\Delta_i) \neq \emptyset$*.*

APPLICATIONS AND RELATED RESULTS

The affine Tverberg theorem was proved by Helge Tverberg in 1966. The topological Tverberg theorem, proved by Bárány, Shlosman, and Szűcs in 1981 (and by Özaydin in 1987 in the prime power case), reduces to the affine version if f is an affine (simplicial) map. Some of the relevant references for these two theorems and their applications are [Bjö95, Sar92, Vol96a, Živ98, Mat02, Mat08].

Note that Theorem 22.4.1 is not a formal consequence of Theorem 22.4.2 since the latter needs an extra condition that r is a prime power. Surprisingly enough this condition turned out to be essential (see Section 22.4.5).

The following "necklace-splitting theorem" of Noga Alon is a very nice application of the continuous Tverberg theorem.

THEOREM 22.4.3 [Alo87]

Assume that an open necklace has ka_i *beads of color* i, $1 \leq i \leq t$, $k \geq 2$*. Then it is possible to cut this necklace at* $t(k-1)$ *places and assemble the resulting intervals into* k *collections, each containing exactly* a_i *beads of color* i*.*

The 'Tverberg-Vrećica conjecture' is a statement that incorporates both the center transversal theorem (Theorem 22.2.4) and the (affine) Tverberg theorem (Theorem 22.4.1) in a single general statement.

CONJECTURE 22.4.4 [TV93]

Assume that $0 \le k \le d-1$ and let $S_0, S_1, \ldots, S_k$ be a collection of finite sets in $\mathbb{R}^d$ of given cardinalities $|S_i| = (r_i - 1)(d-k+1)+1,\ i = 0,1,\ldots,k$. Then S_i can be split into r_i nonempty sets, $S_i^1, \ldots, S_i^{r_i}$, so that for some k-dimensional affine subspace $D \subset \mathbb{R}^d,\ D \cap \mathrm{conv}(S_i^j) \neq \emptyset$ for all i and j, $0 \le i \le k$, $1 \le j \le r_i$.

This conjecture was confirmed in [Živ99] for the case where both d and k are odd integers and $r_i = q$ for each i, where q is an odd prime number, and by Vrećica in the case $r_1 = \ldots = r_k = 2$ [Vre03]. Karasev [Kar07] extended this result to the case where q is a prime power (and arbitrary d and k). Further progress and update on the colored version of this problem can be found in [BMZ11].

The expository article [Kal01] is recommended as a source of additional information about Tverberg-type theorems and conjectures.

22.4.2 COLORED TVERBERG THEOREMS

Let $T(r,k,d)$ be the minimal number t so that for every collection of colors $\mathcal{C} = \{C_0, \ldots, C_k\}$ with the property $|C_i| \ge t$ for all $i = 0, \ldots, k$, there exist r *properly colored* sets $A_i = \{a_j^i\}_{j=0}^k$, $i = 1, \ldots, r$, that are pairwise disjoint but where the corresponding rainbow simplices $\sigma_i := \mathrm{conv}\ A_i$ have a nonempty intersection, $\bigcap_{i=1}^r \sigma_i \neq \emptyset$. A set X is 'properly colored' if it does not contain more than one point of the same color.

The colored Tverberg problem is to establish the existence of, and then to evaluate or estimate, the integer $T = T(r,k,d)$. The cases $k = d$ and $k < d$ are related, but there is also an essential difference. In the case $k = d$, provided t is large enough, the number of intersecting rainbow simplices can be arbitrarily large. In the case $k < d$, for dimension reasons, one cannot expect more than $r \le d/(d-k)$ intersecting k-dimensional rainbow simplices. This is the reason why colored Tverberg theorems are classified as type A or type B, depending on whether $k = d$ or $k < d$. The remaining case $k > d$ is classified as type C.

In the type A case, where $T(r,d,d)$ is abbreviated simply as $T(r,d)$, it is easy to see that a lower bound for this function is r. It is conjectured that this lower bound is attained:

CONJECTURE 22.4.5 (Type A) [BL92]

$T(r,d) = r$ for all r and d.

The colored Tverberg problem (type A) was originally conjectured and designed as a tool for solving important problems of computational geometry [ABFK92, BFL90, BL92] . The weak form of the conjecture, $T(r,d) < +\infty$ [BFL90], is already far from obvious.

Conjecture 22.4.5 is in [BL92] confirmed for $r = 2$ and for $d \le 2$. Živaljević and Vrećica [ŽV92, Živ98, VŽ11] recognized the role of chessboard complexes (Example 22.1.8) as proper configuration spaces for colored Tverberg-type questions. Their central (type A) result [ŽV92] established the bound $T(r,d) \le 2r-1$ if r is a prime power ($T(r,d) \le 4r-3$ in the general case), providing the missing link in the solution of several opened problems in computational geometry (see the end of this section).

Blagojević, Matschke, and Ziegler [BMZ15] observed the importance of type C colored Tverberg questions and applied a similar CS/TM-scheme to established Conjecture 22.4.5 if $r+1$ is a prime number.

THEOREM 22.4.6 (Type A) [BMZ15]

For every integer r such that $r+1$ is a prime number and every collection of $d+1$ disjoint sets ("colors") $C_0, C_1, \dots, C_d$ in $\mathbb{R}^d$, each of cardinality at least r, there exist r disjoint, multicolored subsets $S_i \subset \bigcup_{i=0}^{d} C_i$ such that

$$\bigcap_{i=1}^{r} \operatorname{conv} S_i \neq \emptyset.$$

Recall that in the type B case of the general colored Tverberg problem it is assumed (as a necessary condition) that $r \leq d/(d-k)$. In light of the lower bound $T(r,k,d) \geq 2r-1$ [VŽ94] the following conjecture is quite natural.

CONJECTURE 22.4.7 (Type B)

$T(r,k,d) = 2r-1$.

Here is a theorem confirming Conjecture 22.4.7 if r is a prime power.

THEOREM 22.4.8 (Type B) [VŽ94, Živ98]

Let $C_0, \dots, C_k$ be a collection of $k+1$ disjoint finite sets ("colors") in $\mathbb{R}^d$. Let r be a prime power such that $r \leq d/(d-k)$ and let $|C_i| = t \geq 2r-1$. Then there exist r properly colored k-dimensional simplices S_i, $i = 1, \dots, r$, that are pairwise vertex-disjoint such that

$$\bigcap_{i=1}^{r} \operatorname{conv} S_i \neq \emptyset.$$

REMARK 22.4.9

Both Theorems 22.4.6 and 22.4.8 have their non-linear analogues since they are obtained by a topological argument (see the implications (22.4.8) and (22.4.7). Moreover, as in the case of topological Tverberg theorem, the topological analogue of Theorem 22.4.8 is wrong if r is not a prime power [BFZ15, Remark 4.4].

The type C colored Tverberg results were originally conceived as a tool for the proof of Theorem 22.4.6, however they are certainly interesting in their own right.

THEOREM 22.4.10 (Type C) [BMZ15]

Let $r \geq 2$ be prime, $d \geq 1$, and $N = (r-1)(d+1)$. Let Δ^N be an N-dimensional simplex with a partition of its vertex set into $m+1 \geq d+2$ parts ("color classes"),

$$\operatorname{Vert}(\Delta^N) = C_0 \uplus C_1 \uplus \ldots \uplus C_m,$$

with $|C_i| \leq r-1$ for all i. Then for every continuous map $f : \Delta^N \to \mathbb{R}^d$, there is a collection $F_1, \dots, F_r$ of disjoint faces of Δ^N such that,

(A) $|C_i \cap F_j| \leq 1$ *for each* $i \in \{0, \dots, m\}$ *and* $j \in \{1, \dots, r\}$;

(B) $f(F_1) \cap \ldots \cap f(F_r) \neq \emptyset$.

TOPOLOGICAL BACKGROUND

Suppose that $W_r = \{x \in \mathbb{R}^r \mid x_1 + \ldots + x_r = 0\}$ is the standard $(r-1)$-dimensional real representation of the cyclic group $\mathbb{Z}/r$. The $(r-1)d$-dimensional space $W_r^{\oplus d}$ can be described as the vector space of all real $(r \times d)$-matrices with column sums equal to zero. Let $K^{*r} = K * \ldots * K$ be the join of r copies of K.

Both Theorems 22.4.6 and 22.4.8 follow the CS/TS-scheme based on chessboard complexes $\Delta_{p,r}$ as configuration spaces (Example 22.1.8). The associated *test maps* are respectively (22.4.7) (for Theorem 22.4.8) and (22.4.8) (for Theorem 22.4.6).

$$(\Delta_{r,2r-1})^{*(k+1)} \xrightarrow{\mathbb{Z}/r} W_r^{\oplus d} \tag{22.4.7}$$

$$(\Delta_{r,r-1})^{*d} * [r] \xrightarrow{\mathbb{Z}/r} W_r^{\oplus d}. \tag{22.4.8}$$

Both theorems are consequences of the corresponding Borsuk-Ulam-type statements claiming that in the either case the $\mathbb{Z}/r$-equivariant map must have a zero if r is a prime number.

APPLICATIONS OF COLORED TVERBERG THEOREMS

The bound $T(d+1,d) \le 4d+1$ established in [ŽV92] opened the possibility of proving several interesting results in discrete and computational geometry.

HALVING HYPERPLANES AND THE k-SET PROBLEM

The number $h_d(n)$ of halving hyperplanes of a set of size n in $\mathbb{R}^d$, i.e., the number of essentially distinct placements of a hyperplane that split the set in half, according to Bárány, Füredi, and Lovász [BFL90], satisfies

$$h_d(n) = O(n^{d-\epsilon_d}), \quad \text{where} \quad \epsilon_d = T(d+1,d)^{-(d+1)}.$$

POINT SELECTIONS AND WEAK ϵ-NETS

The equivalence of the following statements was established in [ABFK92] before the inequality $T(d+1,d) < +\infty$ was established in [ŽV92]. Considerable progress has since been made in this area [Mat02], and different combinatorial techniques for proving these statements have emerged in the meantime.

- Weak colored Tverberg theorem: $T(d+1,d)$ is finite.
- Point selection theorem: There exists a constant $s = s_d$, whose value depends on the bound for $T(d+1,d)$, such that any family $\mathcal{H}$ of $(d+1)$-element subsets of a set $X \subset \mathbb{R}^d$ of size $|\mathcal{H}| = p\binom{|X|}{d+1}$ contains a pierceable subfamily $\mathcal{H}'$ such that $|\mathcal{H}'| \gg p^s\binom{|X|}{d+1}$. ($\mathcal{H}'$ is ***pierceable*** if $\bigcap_{S\in\mathcal{H}'} \operatorname{conv} S \ne \emptyset$. $A \gg_d B$ if $A \ge c_1(d)B + c_2(d)$, where $c_1(d) > 0$ and $c_2(d)$ are constants depending only on the dimension d.)
- Weak ϵ-net theorem: For any $X \subset \mathbb{R}^d$ there exists a weak ϵ-net F for convex sets with $|F| \ll_d \epsilon^{(d+1)(1-1/s)}$, where $s = s_d$ is as above. (See Chapter 48 for the notion of ϵ-net; a *weak* ϵ-net is similar, except that it need not be part of X.)

- Hitting set theorem: For every $\eta > 0$ and every $X \subset \mathbb{R}^d$ there exists a set $E \subset \mathbb{R}^d$ that misses at most $\eta\binom{|X|}{d+1}$ simplices of X and has size $|E| \ll_d \eta^{1-s_d}$, where s_d is as above.

OTHER RELATED RESULTS

The configuration space that naturally arises via the CS/TM-scheme in proofs of Theorems 22.4.6 and 22.4.8 is the so-called ***chessboard complex*** $\Delta_{r,t}$, which owes its name to the fact that it can be described as the complex of all non-taking rook placements on an $r \times t$ chessboard. This is an interesting combinatorial object that arises independently as the coset complex of the symmetric group, as the complex of partial matchings in a complete bipartite graph, and as the complex of all partial injective functions. In light of the fact that the high connectivity of a configuration space is a property of central importance for applications (cf. Theorem 22.5.1), chessboard complexes have been studied from this point of view in numerous papers; see [Jon08, VŽ11, JVŽ15] for more recent advances and references.

22.4.3 VAN KAMPEN-FLORES TYPE RESULTS

Classical Van Kampen-Flores theorem [Mat08, Theorem 5.1.1] says that the d-dimensional skeleton $K = (\Delta^{2d+2})^{\leq d}$ of a $(2d+2)$-dimensional simplex Δ^{2d+2} is *strongly non-embeddable* in $\mathbb{R}^{2d}$ in the sense that for each continuous map $f : (\Delta^{2d+2})^{\leq d} \to \mathbb{R}^{2d}$ there exist two vertex disjoint simplices $\sigma_1, \sigma_2 \in K$ such that $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$.

Both the Van Kampen-Flores theorem and its generalized version due to Sarkaria [Sar91], Volovikov [Vol96b], and Blagojević, Frick, and Ziegler [BFZ14], are relatives of the topological Tverberg theorem.

THEOREM 22.4.11 *Generalized Van Kampen–Flores Theorem*

Let $N = (r-1)(d+2)$ where $d \geq 1$ and r is a power of a prime. Let $k \geq \lceil \frac{r-1}{r} d \rceil$. Then for any continuous map $f : \Delta^N \to \mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ^N such that $\dim(\sigma_i) \leq k$ for each i and $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$.

ADMISSIBLE AND PRESCRIBABLE TVERBERG PARTITIONS

In this section we discuss the problem whether each *admissible* r-tuple is *Tverberg prescribable* (or Van Kampan-Flores prescribable). This problem, as formulated in [BFZ14] (see also a related question of R. Bacher, Mathoverflow.net June 2011), will be referred to as the Tverberg A-P problem or the Tverberg A-P conjecture. Following [BFZ14, Definition 6.7] for $d \geq 1$ and $r \geq 2$, we say that an r-tuple $d = (d_1, \dots, d_r)$ of integers is *admissible* if,

$$\lfloor d/2 \rfloor \leq d_i \leq d \qquad \text{and} \qquad \sum_{i=1}^{r} (d - d_i) \leq d. \tag{22.4.9}$$

An admissible r-tuple is *Tverberg prescribable* if there is an N such that for every continuous map $f : \Delta^N \to \mathbb{R}^d$ there is a Tverberg partition $\{\sigma_1, \dots, \sigma_r\}$ for f with $\dim(\sigma_i) = d_i$.

PROBLEM 22.4.12 *Tverberg A-P problem* [BFZ14, Question 6.9.]

Is every admissible r-tuple Tverberg prescribable?

The 'balanced case' of the Tverberg A-P conjecture is the case when the dimensions $d_1, \dots, d_r$ satisfy the condition $|d_i - d_j| \leq 1$ for each i and j. In other words there exist $0 \leq s < r$ and k such that $d_1 = \dots = d_s = k+1$ and $d_{s+1} = \dots = d_r = k$. In this case the second admissibility condition in (22.4.9) reduces to the inequality $rk + s \geq (r-1)d$ while the first condition is redundant. The case when all dimensions are equal $d_1 = \dots = d_r$ is answered by Theorem 22.4.11 (see for example [BFZ14, Theorem 6.5]). The following theorem of D. Jojić, S. Vrećica, and R. Živaljević covers the remaining cases.

THEOREM 22.4.13 *Balanced A-P theorem* [JVŽ15]

Suppose that $r = p^\kappa$ is a prime power and let $\mathbf{d} = (d_1, \dots, d_r)$ be a sequence of integers satisfying the condition $|d_i - d_j| \leq 1$ for each i and j. Then if the sequence $\mathbf{d}$ is admissible then it is Tverberg prescribable.

22.4.4 THE 'CONSTRAINT METHOD'

The Gromov-Blagojević-Frick-Ziegler reduction, or the *'constraint method'*, is an elegant and powerful method for proving results of Tverberg-van Kampen-Flores type. In its basic form the method can be summarized as follows.

$$\begin{array}{ccc} K & \xrightarrow{f} & \mathbb{R}^d \\ {\scriptstyle e}\big\downarrow & & {\scriptstyle i}\big\downarrow \\ \Delta^N & \xrightarrow{F} & \mathbb{R}^{d+1} \end{array} \tag{22.4.10}$$

Suppose that the continuous Tverberg theorem holds for the triple $(\Delta^N, r, \mathbb{R}^{d+1})$ in the sense that for each continuous map $F : \Delta^N \to \mathbb{R}^{d+1}$ there exists a collection of r vertex disjoint faces $\Delta_1, \dots, \Delta_r$ of Δ^N such that $f(\Delta_1) \cap \dots \cap f(\Delta_r) \neq \emptyset$. For example, Theorem 22.4.2 says that this is the case if $r = p^k$ is a prime power and $N = (r-1)(d+2)$. Suppose that $K \subset \Delta^N$ is a simplicial complex which is *r-unavoidable* in the sense that if $A_1 \uplus \dots \uplus A_r = [N+1]$ is a partition of the set $[N+1]$ (of vertices of Δ), then at least one of the faces $\Delta(A_i)$ of Δ^N (spanned by A_i) is in K. Then for each continuous map $f : K \to \mathbb{R}^d$ there exists vertex disjoint simplices $\sigma_1, \dots, \sigma_r \in K$ such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$.

Indeed, let $\bar{f}$ be an extension ($\bar{f} \circ e = f$) of the map f to Δ^N. Suppose that $\rho : \Delta^N \to \mathbb{R}$ is the function $\rho(x) := \mathrm{dist}(x, K)$, measuring the distance of the point $x \in \Delta^N$ from K. Define $F = (\bar{f}, \rho) : \Delta^N \to \mathbb{R}^{d+1}$ and assume that $\Delta_1, \dots, \Delta_r$ is the associated family of vertex disjoint faces of Δ^N, such that $F(\Delta_1) \cap \dots \cap F(\Delta_r) \neq \emptyset$. More explicitly suppose that $x_i \in \Delta_i$ such that $F(x_i) = F(x_j)$ for each $i, j = 1, \dots, r$. Since K is r-unavoidable, $\Delta_i \in K$ for some i. As a consequence $\rho(x_i) = 0$, and in turn $\rho(x_j) = 0$ for each $j = 1, \dots, r$. If Δ_i' is the minimal face of Δ^N containing x_i then $\Delta_i' \in K$ for each $i = 1, \dots, r$ and $f(\Delta_1') \cap \dots \cap f(\Delta_r') \neq \emptyset$.

For a more complete exposition and numerous examples of applications of the 'constraint method' the reader is referred to [BFZ14], see also [Gro10, Section 2.9(c)] and [Lon02, Proposition 2.5].

22.4.5 COUNTEREXAMPLES

In many results of Tverberg-van Kampen-Flores type there is a conspicuous condition that the number of intersecting simplices is a power of a prime $r = p^k$ (see Theorems 22.4.2, 22.4.8, 22.4.11). It is probably safe to say that a majority of specialists in the area believed that this condition is not essential, and that it will be eventually removed from these statements (note its absence in the formulation of the Affine Tverberg Theorem, Thereom 22.4.1).

It was a deep insight of Isaac Mabillard and Uli Wagner that the truth may be quite the opposite. Motivated by the intriguing absence of counterexamples in problems of Tverberg-Van Kampen type, they initiated in [MW14] the program of studying maps $f : K \to \mathbb{R}^d$ without triple, quadruple, or, more generally, global r-fold points (r-Tverberg points).

A necessary condition for such a map to exist is the existence of an S_r-equivariant map $F : K^{\times r}_{\Delta} \to S(W_r^{\oplus d})$ where S_r is the symmetric group, $W_r = \{x \in \mathbb{R}^d \mid x_1 + \ldots + x_r = 0\}$, and $K^{\times r}_{\Delta}$ is the associated r-*fold deleted product* of K defined as the union of all products $\sigma_1 \times \cdots \times \sigma_r$ of simplices such that $\sigma_i \in K$ for all i and $\sigma_i \cap \sigma_j \neq \emptyset$ for each $i \neq j$.

The following remarkable result was the first in a row of theorems which paved the way for long awaiting counterexamples in this area. The reader is referred to [MW14, MW16a, AMSW15, MW16b] for a complete exposition of the theory and subsequent developments.

THEOREM 22.4.14 (I. Mabillard, U. Wagner [MW14, MW16a])

Suppose that $r \geq 2, k \geq 3$, and let K be a simplicial complex of dimension $(r-1)k$. Then the following statements are equivalent:

(i) *There exists an S_r-equivariant map $F : K^{\times r}_{\Delta} \to S(W_r^{\oplus rk})$.*

(ii) *There exists a continuous map $f : K \to \mathbb{R}^{rk}$ such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset$ for each collection of pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of K.*

It has been known for quite some time that the statement (i) in Theorem 22.4.14 is satisfied if r is *not* a prime power (M. Özaydin [Öza87]). It follows that Theorem 22.4.14 alone is strong enough to provide counterexamples for van Kampen type questions in the non prime power case. For example it implies that Theorem 22.4.11 is in general false, unless r is a prime power.

As far as the topological Tverberg theorem (Theorem 22.4.2) is concerned Theorem 22.4.14 was not strong enough for this purpose. A new idea was needed and it came in the form of the Gromov-Blagojević-Frick-Ziegler reduction (Section 22.4.4). Surprisingly enough, this idea was introduced before the appearance of [MW14] and Theorem 22.4.14.

Gromov studied in [Gro10] the lower bounds on the topological complexity of the fibers $F^{-1}(y) \subset X$ of continuous maps $F : X \to Y$, in terms of combinatorial and topological invariants of spaces X and Y. The statement *"The topological Tverberg theorem, whenever available, implies the van Kampen-Flores theorem"* appears in Section 2.9(c), with a short proof.

Unaware of [Gro10], Blagojević, Frick, and Ziegler, developed in [BFZ14] their 'constraint method' as a powerful tool for reducing various Tverberg type statements to the original topological Tverberg theorem.

The following result was announced in [Fri15], see also [BFZ15].

THEOREM 22.4.15 (Frick [Fri15], Blagojević, Frick, and Ziegler [BFZ15])
Assume that $r \geq 6$ is an integer that is not a prime power, let $k \geq 3$ and $N = (r-1)(rk+2)$. Then there exists a continuous map $f : \Delta^N \to \mathbb{R}^{rk+1}$ such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset$ for each collection of pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of Δ^N.

For a more complete exposition, refinements and generalizations of these results the reader is referred to [AMSW15, MW14, MW16a, BFZ15].

22.5 TOOLS FROM EQUIVARIANT TOPOLOGY

The method of equivariant maps is a versatile tool for proving results in discrete geometry and combinatorics. For many results these are the only proofs available. Equivariant maps are typically encountered at the final stage of application of the CS/TM-scheme (Section 22.1).

GLOSSARY

G-space X, G-action: A group G acts on a space X if each element of G is a continuous transformation of X and multiplication in G corresponds to composition of transformations. Formally, a G-action α is a continuous map $\alpha : G \times X \to X$ such that $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$. Then X is called a G-space and $\alpha(g, x)$ is often abbreviated as $g \cdot x$ or gx.

Free G-action: A G-action is free if $g \cdot x = x$ for some $x \in X$ implies $g = e$, where e is the unit element in G. An action is fixed-point free if for each $x \in X$ there exists $g \in G$ such that $g \cdot x \neq x$.

G-equivariant map: A map $f : X \to Y$ of two G-spaces X and Y is equivariant if for all $g \in G$ and $x \in X$, $f(g \cdot x) = g \cdot f(x)$.

Borsuk-Ulam-type theorem: Any theorem establishing the nonexistence of a G-equivariant map between two G-spaces X and Y.

n-Connected space: A path-connected and simply connected space with trivial homology in dimensions $1, 2, \ldots, n$. A path-connected space X is simply connected or 1-connected if every closed loop $\omega : S^1 \to X$ can be deformed to a point.

The following generalization of the Borsuk-Ulam theorem is the key result used in many proofs. Note that if $X = S^n$, $Y = S^{n-1}$, and $G = \mathbb{Z}_2$, it specializes to a statement equivalent to the standard version of the Borsuk-Ulam theorem (Theorem 22.2.2).

THEOREM 22.5.1 [Dol83]
Suppose X and Y are simplicial (more generally CW) complexes equipped with the free action of a finite group G, and that X is m-connected, where $m = \dim Y$. Then there does not exist a G-equivariant map $f : X \to Y$.

The following refinement of Theorem 22.5.1 (due to A.Yu. Volovikov) allows us to treat some interesting cases of non-free G-actions.

THEOREM 22.5.2 [Vol96a]

Let p be a prime number and $G = (\mathbb{Z}_p)^k$ an elementary abelian p-group. Suppose that X and Y are fixed-point free G-spaces such that $\widetilde{H}^i(X, \mathbb{Z}_p) \cong 0$ for all $i \leq n$ and Y is an n-dimensional cohomology sphere over $\mathbb{Z}_p$. Then there does not exist a G-equivariant map $f : X \to Y$.

A topological index theory is a complexity theory for G-spaces that allows us to conclude that there does not exist a G-equivariant map $f : X \to Y$ if the G-space X is of *larger complexity* than the G-space Y. A measure of complexity of a given G-space is the so-called *equivariant index* $\mathrm{Ind}_G(X)$. In general, an index function is defined on a class of G-spaces, say all finite G-CW complexes, and takes values in a suitable partially ordered set Ω. For example if $G = \mathbb{Z}_2$, an index function $\mathrm{Ind}_{\mathbb{Z}_2}(X)$ is defined as the minimum integer n such that there exists a $\mathbb{Z}_2$-equivariant map $f : X \to S^n$. In this case $\Omega := \mathbb{N}$ is the poset of nonnegative integers. Note that the Borsuk-Ulam theorem simply states that $\mathrm{Ind}_{\mathbb{Z}_2}(S^n) = n$.

PROPOSITION 22.5.3 [Mat08, Živ98]

For each nontrivial finite group G, there exists an integer-valued index function $\mathrm{Ind}_G(\cdot)$ *defined on the class of finite, G-simplicial complexes such that*

(i) *If* $\mathrm{Ind}_G(X) > \mathrm{Ind}_G(Y)$, *then a G-equivariant map $f : X \to Y$ does not exist.*

(ii) *If X is $(n-1)$-connected then* $\mathrm{Ind}_G(X) \geq n$.

(iii) *If X is an n-dimensional, free G-complex then* $\mathrm{Ind}_G(X) \leq n$.

(iv) $\mathrm{Ind}_G(X * Y) \leq \mathrm{Ind}_G(X) + \mathrm{Ind}_G(Y) + 1$, *where $X * Y$ is the* join *of spaces.*

It is clear that the computation or good estimates of the complexity indices $\mathrm{Ind}_G(X)$ are essential for applications. Occasionally this can be done even if the details of construction of the index function are not known. Such a tool for finding the lower bounds for an index function described in Proposition 22.5.2 is provided by the following inequality.

PROPOSITION 22.5.4 *Sarkaria inequality* [Mat08, Živ98]

Let L be a free G-complex and $L_0 \subset L$ a G-invariant, simplicial subcomplex. Let $\Delta(L \setminus L_0)$ be the order complex (cf. Chapter 21) *of the complementary poset $L \setminus L_0$. Then*

$$\mathrm{Ind}_G(L_0) \geq \mathrm{Ind}_G(L) - \mathrm{Ind}_G(\Delta(L \setminus L_0)) - 1.$$

The index function described Proposition 22.5.3 in its basic form relies on Proposition 22.5.1 and can be applied only to free group actions. A more general index function which has all the expected properties (including the Sarkaria's inequality) and which can be applied to some non-free group actions is described in [JVŽ16]. In some applications it is more natural, and sometimes essential, to use more sophisticated partially ordered sets of G-degrees of complexity. A notable example is the *ideal valued index theory* of S. Husseini and E. Fadell [FH88], which proved useful in establishing the existence of equilibrium points in incomplete markets (mathematical economics).

A BRIEF GUIDE TO THE LITERATURE

The book [Mat08] is an excellent introduction and a guide to other applications of topological methods, including the graph coloring problems (Kneser's conjecture). The monograph [Die87] provides a lot of foundational material and introduces the reader into the more advanced topics in the theory of transformation groups. The reader interested in applications to combinatorics and discrete geometry will find here both a detailed exposition of equivariant obstruction theory (Section II.3) and some very interesting applications (the Hopf classification theorem, the classification of equivariant maps between representation spheres, etc.).

The problem of the existence of G-equivariant maps $f : X \to Y$ (in the case when Y is a sphere) is closely related to the problem of finding a non-zero section of a vector bundle [Die87, Section I.7]. A far reaching 'singularity approach' to this problem is described in the monograph [Kos81]. This approach can be quite effective, especially in the case when it is sufficient to calculate the first obstruction, see [Gro03, Section 5.1], [KHA14, Section 5.3], and [VŽ15, Section 2.4] for some illustrative examples.

RELATED CHAPTERS

Chapter 1: Finite point configurations
Chapter 4: Helly-type theorems and geometric transversals
Chapter 24: Embedding and geometric realization
Chapter 68: Applications to structural molecular biology

REFERENCES

[ABFK92] N. Alon, I. Bárány, Z. Füredi, and D. Kleitman. Point selections and weak ϵ-nets for convex hulls. *Combin. Probab. Comput.*, 1:189-200, 1992.

[AET00] N. Amenta, D. Eppstein, and S-H. Teng. Regression depth and center points. *Discrete Comput. Geom.*, 23:305–329, 2000.

[Alo87] N. Alon. Splitting necklaces. *Adv. Math.*, 63:247–253, 1987.

[AMSW15] S. Avvakumov, I. Mabillard, A. Skopenkov, and U. Wagner. Eliminating higher-multiplicity intersections, III. Codimension 2. Preprint, `arXiv:1511.03501`, 2015.

[BB89] I.K. Babenko and S.A. Bogatyi. On the mapping of a sphere into Euclidean space (Russian). *Mat. Zametki*, 46:3–8, 1989; translated in *Math. Notes*, 46:683–686, 1989.

[BFHZ15] P.V.M. Blagojević, F. Frick, A. Haase, and G.M. Ziegler. Topology of the Grünbaum-Hadwiger-Ramos hyperplane mass partition problem. Preprint, `arXiv:1502.02975`, 2015.

[BFHZ15] P.V.M. Blagojević, F. Frick, A. Haase, and G.M. Ziegler. Hyperplane mass partitions via relative equivariant obstruction theory. Preprint, `arXiv:1509.02959`, 2015.

[BFL90] I. Bárány, Z. Füredi, and L. Lovász. On the number of halving planes. *Combinatorica*, 10:175–183, 1990.

[BFZ14] P.V.M. Blagojević, F. Frick, and G.M. Ziegler. Tverberg plus constraints. *B. London Math. Soc.*, 46:953–967, 2014.

[BFZ15] P.V.M. Blagojević, F. Frick, and G.M. Ziegler. Barycenters of Polytope Skeleta and Counterexamples to the Topological Tverberg Conjecture, via Constraints. Preprint, `arXiv:1510.07984`, 2015.

[Bjö95] A. Björner. Topological methods. In R. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*, pages 1819–1872, North-Holland, Amsterdam, 1995.

[BK14] B. Bukh and R.N. Karasev. Suborbits in Knaster's problem. *Bull. London Math. Soc.*, 46:269-278, 2014.

[BL92] I. Bárány and D.G. Larman. A colored version of Tverberg's theorem. *J. London Math. Soc.*, 45:314–320, 1992.

[BLVŽ94] A. Björner, L. Lovász, S.T. Vrećica, and R.T. Živaljević. Chessboard complexes and matching complexes. *J. London Math. Soc. (2)*, 49:25–39, 1994.

[BM01] I. Bárány and J. Matoušek. Simultaneous partitions of measures by k-fans, *Discrete Comput. Geom.*, 25:317–334, 2001.

[BM02] I. Bárány and J. Matoušek. Equipartitions of two measures by a 4-fan. *Discrete Comput. Geom.*, 27:293–301, 2002.

[BMZ11] P.V.M. Blagojević, B. Matschke, and G.M. Ziegler. Optimal bounds for a colorful Tverberg-Vrećica type problem. *Adv. Math.*, 226:5198-5215, 2011.

[BMZ15] P.V.M. Blagojević, B. Matschke, and G.M. Ziegler. Optimal bounds for the colored Tverberg problem. *J. Eur. Math. Soc.*, 17:739–754, 2015.

[Bor33] K. Borsuk. Drei Sätze über die n-dimensionale euklidische Sphäre. *Fund. Math.*, 20:177–190, 1933.

[BP15] V.M. Buchstaber and T.E. Panov, *Toric Topology.* Vol. 204 of *Math. Surveys Monog.*, AMS, Providence, 2015,

[Bre93] G.E. Bredon. *Topology and Geometry*, Vol. 139 of *Graduate Texts in Math.*, Springer, New York, 1993.

[Bro12] L.E.J. Brouwer. Über Abbildung von Mannigfaltigkeiten. *Math. Ann.*, 71:97–115, 1912.

[BSS81] I. Bárány, S.B. Shlosman, and A. Szűcs. On a topological generalization of a theorem of Tverberg. *J. London Math. Soc.*, 23:158–164, 1981.

[BT95] S.J. Brams and A.D. Taylor. An envy-free cake division protocol. *Amer. Math. Monthly*, 102:9-18, 1995.

[BZ14] P.V.M. Blagojević and G. M. Ziegler. Convex equipartitions via equivariant obstruction theory. *Israel J. Math.*, 200:49-77, 2014.

[Die87] T. tom Dieck. *Transformation Groups*. Vol. 8 of *Studies in Mathematics*, Walter de Gruyter, Berlin, 1987.

[Dol83] A. Dold. Simple proofs of some Borsuk-Ulam results. *Contemp. Math.*, 19:65–69, 1983.

[Dol93] V.L. Dol'nikov. Transversals of families of sets in $\mathbb{R}^n$ and a relationship between Helly and Borsuk theorems. *Mat. Sb.*, 184:111–131, 1993.

[FH88] E. Fadell and S. Husseini. An ideal-valued cohomological index theory with applications to Borsuk-Ulam and Bourgin-Yang theorems. *Ergodic Theory Dynam. Systems*, 8^*:73–85, 1988.

[Fri15] F. Frick. Counterexamples to the topological Tverberg conjecture. Preprint, `arXiv:1502.00947`, 2015.

[Gro03] M. Gromov. Isoperimetry of waists and concentration of maps. *Geom. Funct. Anal.,* 13:178–215, 2003.

[Gro10] M. Gromov. Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry. *Geom. Funct. Anal.*, 20:416-526, 2010.

[HMS02] T. Hausel, E. Makai, Jr., and A. Szűcs. Inscribing cubes and covering by rhombic dodecahedra via equivariant topology. *Mathematika*, 47:371–397, 2002.

[Jon08] J. Jonsson. *Simplicial Complexes of Graphs*. Vol. 1928 of *Lecture Notes in Math.*, Springer, Heidelberg, 2008.

[JVŽ15] D. Jojić, S. Vrećica, and R. Živaljevic. Symmetric multiple chessboard complexes and a new theorem of Tverberg type. Preprint, `arXiv:1502.05290`, 2015.

[JVŽ16] D. Jojić, S. Vrećica, and R. Živaljevic. Topology and combinatorics of 'unavoidable complexes.' Preprint, `arXiv:1603.08472`, 2016.

[KK93] J. Kahn and G. Kalai. A counterexample to Borsuk's conjecture. *Bull. Amer. Math. Soc.*, 29:60–62, 1993.

[Kal01] G. Kalai. Combinatorics with a geometric flavor. In N. Alon, J. Bourgain, A. Connes, M. Gromov, and V. Milman, editors, *Visions in Mathematics: GAFA 2000 Special volume, Part II.*, pages 742–791, Birkhäuser, Basel, 2001.

[Kar07] R.N. Karasev. Tverbergs transversal conjecture and analogues of nonembeddability theorems for transversals. *Discrete Comput. Geom.*, 38:513-525 (2007).

[Kar09] R.N. Karasev. Dual theorems on central points and their generalizations. *Sbornik. Mathematics*, 199:1459–1479, 2008.

[KHA14] R. N. Karasev, A. Hubard, and B. Aronov. Convex equipartitions: the spicy chicken theorem. *Geom. Dedicata*, 170:263-279, 2014.

[KK99] A. Kaneko and M. Kano. Balanced partitions of two sets of points in the plane. *Comput. Geom.*, 13:253–261, 1999.

[KM14] R.N. Karasev and B. Matschke. Projective center point and Tverberg theorems. *Discrete Comput. Geom.*, 52:88–101, 2014.

[KRPS16] R.N. Karasev, E. Roldán-Pensado, and P. Soberón. Measure partitions using hyperplanes with fixed directions. *Israel J. Math.*, 212:705–728, 2016.

[Kos81] U. Koschorke, *Vector Fields and Other Vector Bundle Morphisms: A Singularity Approach*. Vol. 847 of *Lecture Notes in Math.*, Springer, Heidelberg, 1981.

[Kup99] G. Kuperberg. Circumscribing constant-width bodies with polytopes. *New York J. Math.* 5:91–100, 1999.

[Lon02] M. de Longueville. Notes on the topological Tverberg theorem, *Discrete Math.*, 241:207-233, 2001. Erratum: *Discrete Math.*, 247:271-297, 2002.

[LŽ08] M. de Longueville and R.T. Živaljević. Splitting multidimensional necklaces. *Adv. Math.*, 218:926–939, 2008.

[Mak86] V.V. Makeev. Some properties of continuous mappings of spheres and problems in combinatorial geometry. In L.D. Ivanov, editor, *Geometric Questions in the Theory of Functions and Sets* (Russian), pages 75–85, Kalinin State Univ., 1986.

[Mak90] V.V. Makeev. The Knaster problem on the continuous mappings from a sphere to a Euclidean space. *J. Soviet Math.*, 52:2854–2860, 1990.

[Mak94] V.V. Makeev. Inscribed and circumscribed polygons of a convex body. *Mat. Zametki*, 55:128–130, 1994; translated in *Math. Notes*, 55:423–425, 1994.

[Mak98] V.V. Makeev. Some special configurations of planes that are associated with convex compacta (Russian). *Zap. Nauchn. Sem. S.-Petersburg* (POMI), 252:165–174, 1998.

[Mak01] V.V. Makeev. Equipartition of a mass continuously distributed on a sphere and in space (Russian). *Zap. Nauchn. Sem. S.-Petersburg* (POMI), 279:187–196, 2001.

[Mat02] J. Matoušek. *Lectures on Discrete Geometry.* Volume 212 of *Graduate Texts in Math.*, Springer-Verlag, New York, 2002.

[Mat08] J. Matoušek. *Using the Borsuk-Ulam Theorem. Lectures on Topological Methods in Combinatorics and Geometry.* Second edition, Universitext, Springer, Heidelberg, 2008.

[MTTV97] G.L. Miller, S.-H. Teng, W. Thurston, and S. Vavasis. Separators for sphere-packings and nearest neighbor graphs. *J. ACM*, 44:1–29, 1997.

[MTTV98] G.L. Miller, S.-H. Teng, W. Thurston, S.A. Vavasis. Geometric separators for finite-element meshes. *SIAM J. Sci. Comput.*, 19:364–386, 1998.

[Mun84] J.R. Munkres. *Elements of Algebraic Topology.* Addison-Wesley, Menlo-Park, 1984.

[MVŽ01] E. Makai, S. Vrećica, and R. Živaljević. Plane sections of convex bodies of maximal volume. *Discrete Comput. Geom.*, 25:33–49, 2001.

[MW14] I. Mabillard and U. Wagner. Eliminating Tverberg points, I. An analogue of the Whitney trick. In *Proc. 30th Sympos. Comput. Geom.* pages. 171–180, ACM Press, 2014.

[MW16a] I. Mabillard and U. Wagner. Eliminating higher-multiplicity intersections, I. a Whitney trick for Tverberg-type problems. In *proc. 32nd Sympos. Comput. Geom.*, pages 51:1–51:12, vol. 51 of *LIPIcs*, Schloss Dagstuhl, 2016.

[MW16b] I. Mabillard and U. Wagner. Eliminating higher-multiplicity intersections, II. The deleted product criterion in the r-metastable range. Preprint, `arXiv:1601.00876`, 2016.

[Öza87] M. Özaydin. Equivariant maps for the symmetric group. Unpublished manuscript, available online at `http://minds.wisconsin.edu/handle/1793/63829`, 1987.

[Rad46] R. Rado. Theorem on general measure. *J. London Math. Soc.*, 21:291–300, 1946.

[Ram96] E. Ramos. Equipartitions of mass distributions by hyperplanes. *Discrete Comput. Geom.*, 15:147–167, 1996.

[Sar91] K.S. Sarkaria, A generalized van KampenFlores theorem. *Proc. Amer. Math. Soc.* 11:559-565, 1991.

[Sar92] K.S. Sarkaria. Tverberg's theorem via number fields. *Israel J. Math.*, 79:317–320, 1992.

[Sob12] P. Soberón. Balanced convex partitions of measures in $\mathbb{R}^d$. *Mathematika*, 58:71-76, 2012.

[Ste85] H. Steinlein. Borsuk's antipodal theorem and its generalizations and applications: a survey. In *Topological Methods in Nonlinear Analysis*, volume 95 of *Sém. Math. Sup.*, pages 166–235, Presses de l'Université de Montréal, 1985.

[Tve66] H. Tverberg. A generalization of Radon's theorem. *J. London Math. Soc.*, 41:123–128, 1966.

[TV93] H. Tverberg and S. Vrećica. On generalizations of Radon's theorem and the ham sandwich theorem. *Europ. J. Combin.*, 14:259–264, 1993.

[Vol96a] A.Y. Volovikov. On a topological generalization of the Tverberg theorem. *Math. Notes*, 59:324–32, 1996.

[Vol96b] A.Y. Volovikov. On the van Kampen-Flores theorem, *Math. Notes*, 59:477-481, 1996.

[Vre03] S. Vrećica. Tverberg's conjecture. *Discrete Comput. Geom.*, 29:505–510, 2003.

[Vre09] S. Vrećica. Equipartitions of a mass in boxes. *J. Combin. Theory Ser. A*, 116:132–142, 2009.

[VŽ92] S. Vrećica and R. Živaljević. The ham sandwich theorem revisited. *Israel J. Math.*, 78:21–32, 1992.

[VŽ94] S. Vrećica and R. Živaljević. New cases of the colored Tverberg theorem. In H. Barcelo and G. Kalai, editors, *Jerusalem Combinatorics'93*, pages 325–334. Volume 178 of *Contemp. Math.*, AMS, Providence, 1994.

[VŽ11] S. Vrećica and R. Živaljević. Chessboard complexes indomitable. *J. Combin. Theory Ser. A*, 118:2157–2166, 2011.

[VŽ15] S. Vrećica and R. Živaljević. Hyperplane mass equipartition problem and the shielding functions of Ramos. Preprint, `arXiv:1508.01552`, 2015.

[Zie15] G.M. Ziegler. Cannons at sparrows. *Eur. Math. Soc. Newsl.*, 95:25–31, 2015.

[Živ98] R. Živaljević. User's guide to equivariant methods in combinatorics, I and II. *Publ. Inst. Math. (Beograd) (N.S.)*, (I) 59:114–130, 1996 and (II) 64:107–132, 1998.

[Živ99] R. Živaljević. The Tverberg-Vrećica problem and the combinatorial geometry on vector bundles. *Israel. J. Math.*, 111:53–76, 1999.

[Živ15] R. Živaljević. Computational topology of equipartitions by hyperplanes. *Topol. Methods Nonlinear Anal.*, 45:63–90, 2015.

[ŽV90] R. Živaljević and S. Vrećica. An extension of the ham sandwich theorem. *Bull. London Math. Soc.*, 22:183–186, 1990.

[ŽV92] R.T. Živaljević and S.T. Vrećica. The colored Tverberg's problem and complexes of injective functions. *J. Combin. Theory Ser. A*, 61:309–318, 1992.