14 TOPOLOGICAL METHODS

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INTRODUCTION

A problem is solved or some other goal achieved by "topological methods" if in our arguments we appeal to the "form," the "shape," the "global" rather than "local" structure of the object or configuration space associated with the phenomenon we are interested in. This configuration space is typically a manifold or a simplicial complex. The global properties of the configuration space are usually expressed in terms of its homology and homotopy groups, which capture the idea of the higher (dis)connectivity of a geometric object and to some extent provide *an analysis properly geometric or linear that expresses location directly as algebra expresses magnitude.*[1]

Thesis: *Any global effect that depends on the object as a whole and that cannot be localized is of homological nature, and should be amenable to topological methods.*

WHERE HAS TOPOLOGY BEEN APPLIED IN COMPUTER SCIENCE?

The references [ATCS] and [BEA+99] provide a broad overview of many current applications of algebraic topology in computer science and vice versa as well as an insight into promising new developments. The field is undergoing a rapid expansion and the following list should be understood as a sample of some of the main themes or aspects of potential future research.

(a) Algebraic topology (AT) is viewed as a useful tool in solving combinatorial or discrete geometric problems of relevance to computing and the analysis of algorithms, [Mat02, Mata, Živ98].

(b) *Computational topology* emerges [BEA+99] as a separate branch of computational geometry unifying topological questions in computer applications such as image processing, cartography, computer graphics, solid modeling, mesh generation, and molecular modeling [BEA+99, DEG99].

(c) Effective algebraic topology deals with algorithmic and computational aspects of topology including the recognition problem (3-manifolds), effective computations of topological invariants (homology, homotopy groups, knot invariants), etc. [Ser].

(d) Combinatorial proofs of statements originally obtained by nonconstructive topological methods were discovered [Matb, Zie02].

(e) The methods of AT can provide qualitative and shape information unavailable by the use of other methods. For example AT provides a tool for visualization

[1] A dream of G.W. Leibniz expressed in a letter to C. Huygens dated 1697; see [Bre95, Chap. 7].

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and feature identification in highly complex empirical data, e.g., in biogeometry [BioG].

(f) AT provides a useful framework for analyzing problems in distributed and concurrent computing [HR95, HR00].

HOW IS TOPOLOGY APPLIED IN DISCRETE GEOMETRIC PROBLEMS?

In this chapter we put some emphasis on the role of (equivariant) topological methods in solving combinatorial or discrete geometric problems that have proven to be of relevance for computational geometry and computational mathematics in general. The versatile *configuration space/test map* scheme was developed in numerous research papers over the years and formally codified in [Živ98]. Its essential features are the following two steps:

Step 1: The problem is rephrased in topological terms.

The problem should give us a clue how to define a "natural" *configuration space* X and how to rephrase the question in terms of zeros or coincidences of the associated *test maps*. Alternatively the problem may be divided into several subproblems, in which case one is often led to the question of when the subsets of X corresponding to the various subproblems have *nonempty intersection*.

Step 2: A standard topological technique is used to solve the rephrased problem.

The topological technique that is most frequently used in discrete geometric problems is based on the technique of *intersecting homology classes* and on *generalized Borsuk-Ulam theorems.*

14.1 THE CONFIGURATION SPACE/TEST MAP PARADIGM

GLOSSARY

Configuration space/test map scheme (CS/TM): A very useful and general scheme for proving combinatorial or geometric facts. The problem is reduced to the question of showing that there does not exist a G-equivariant map $f : X \to V \setminus Z$ (Section 14.5) where X is the configuration space, V the test space, and Z the test subspace associated with the problem, while G is a naturally arising group of symmetries.

Configuration space: In general, any topological space X that parameterizes a class of configurations of geometric objects (e.g., arrangements of points, lines, fans, flags, etc.) or combinatorial structures (trees, graphs, partitions, etc.). Given a problem $\mathcal{P}$, an associated configuration or ***candidate space*** $X_{\mathcal{P}}$ collects all geometric configurations that are (reasonable) candidates for a solution of $\mathcal{P}$.

Test map and ***test space :*** A map $t : X_{\mathcal{P}} \to V$ from the configuration space $X_{\mathcal{P}}$ into the so-called test space V that tests the validity of a candidate $p \in X_{\mathcal{P}}$ as a solution of $\mathcal{P}$. The final ingredient is the ***test subspace*** $Z \subset V$, where $p \in X$ is a solution to the problem if and only if $t(p) \in Z$. Usually $V \cong \mathbb{R}^d$ while Z is just the origin $\{0\} \subset V$ or more generally a linear subspace arrangement in V.

Equivariant maps: The final ingredient in the CS/TM-scheme is a group G of symmetries that acts on both the configuration space $X_{\mathcal{P}}$ and the test space V (keeping the test subspace Z invariant). The test map t is always assumed G-equivariant, i.e., $t(g \cdot x) = g \cdot t(x)$ for each $g \in G$ and $x \in X_{\mathcal{P}}$. Some of the methods and tools of equivariant topology are outlined in Section 14.5.

EXAMPLE 14.1.1 (Y. Soibelman [Soi02])

Suppose that ρ is a metric on $\mathbb{R}^2$ that induces the same topology as the usual Euclidean metric. In other words we assume that for each sequence of points $(x_n)_{n\geq 0}$, $\rho(x_n, x_0) \to 0$ if and only if $|x_n - x_0| \to 0$. Then there exists a ρ-equilateral triangle, i.e., a triple (a, b, c) of distinct points in $\mathbb{R}^2$ such that $\rho(a, b) = \rho(b, c) = \rho(c, a)$.

This is our first example that illustrates the CS/TM-scheme. The configuration space X should collect all candidates for the solution, so a first, "naive" choice is the space of all (ordered) triples $(x, y, z) \in \mathbb{R}^2$. Of course we can immediately rule out some obvious nonsolutions, e.g., degenerate triangles (x, y, z) such that at least one of numbers $\rho(x, y), \rho(y, z), \rho(z, x)$ is zero. (This illustrates the fact that in general there may be several possible choices for a configuration space associated to the initial problem.) Our choice is $X := \mathbb{R}^2 \setminus \Delta$ where $\Delta := \{(x, x, x) \mid x \in \mathbb{R}^2\}$. A "triangle" $(x, y, z) \in X$ is ρ-equilateral if and only if $(\rho(x, y), \rho(y, z), \rho(z, x)) \in Z$, where $Z := \{(u, u, u) \in \mathbb{R}^3 \mid u \in \mathbb{R}\}$. Hence a test map $t : X \to \mathbb{R}^3$ is defined by $t(x, y, z) = (\rho(x, y), \rho(y, z), \rho(z, x))$, the test space is $V = \mathbb{R}^3$, and $Z \subset \mathbb{R}^3$ is the associated test subspace. A triangle $\{x, y, z\}$, viewed as a set of vertices, is in general labeled by six different triples in the configuration space X. This redundancy is a motivation for introducing the group of symmetries $G = S_3$, which acts on both the configuration space X and the test space V. The test map t is clearly S_3-equivariant. If the image of t is disjoint from Z, there arises an S_3-equivariant map from X to $V \setminus Z$. If S^1 is the unit circle in a 2-plane in $V = \mathbb{R}^3$ orthogonal to $Z \cong \mathbb{R}^1$, then projection and normalization give an S_3-equivariant map $\alpha : V \setminus Z \to S^1$. The unit 3-sphere S^3 in a 4-plane orthogonal to Δ is S_3-invariant, hence the inclusion map $\beta : S^3 \to X$ is S_3-equivariant. Finally, the composition $f = \beta \circ t \circ \alpha : S^3 \to S^1$ is also S_3-equivariant, which leads to a contradiction. One way to prove this is to use Theorem 14.5.1, since the sphere S^3 is clearly 1-connected.

Here is another example of how topology comes into play and proves useful in geometric and combinatorial problems. The *configuration space* associated to the next problem is a 2-dimensional torus $T^2 \cong S^1 \times S^1$. This time, however, the test map is not explicitly given. Instead, the problem is reduced to counting intersection points of two "test subspaces" in T^2.

EXAMPLE 14.1.2 *A watch with two equal hands*

A watch was manufactured with a defect so that both hands (minute and hour) are identical. Otherwise the watch works well and the question is to determine the number of ambiguous positions, i.e., the positions for which it is not possible to determine the exact time.

First of all we observe that every position of a hand is determined by an angle

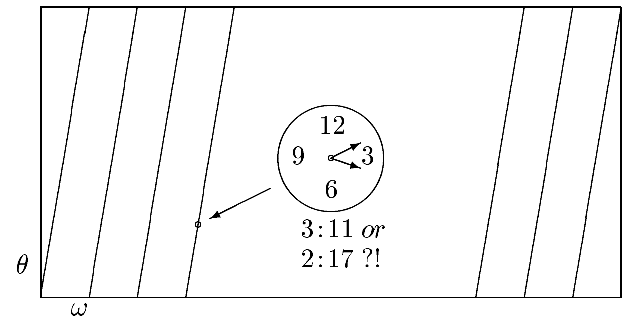

FIGURE 14.1.1
The configuration space of the two hands is a torus

$\omega \in [0, 2\pi]$, so that the configuration space of all possible positions of a hand is homeomorphic to the unit circle S^1. Two independent hands have the 2-dimensional torus $T^2 \cong S^1 \times S^1$ as their configuration space, i.e., the space representing all allowed states or positions of the system. A usual model of a torus is a square or a rectangle (see Figure 14.1.1) with the opposite sides glued together. If θ corresponds to the minute hand and ω is the coordinate of the hour hand, then the fact that the first hand is twelve times faster is recorded by the equation $\theta = 12\,\omega$. This equation describes a curve Γ_1 on the torus T^2, which is just a circle winding 12 times in the direction of the θ axis while it winds only once in the direction of ω axis. The curve Γ_1 is represented in our picture as the union of 12 line segments, seven of them indicated in Figure 14.1.1. If the hands change places then the corresponding curve Γ_2 has equation $\omega = 12\,\theta$. The ambiguous positions are exactly the intersection points of these two curves (except those that belong to the diagonal $\Delta := \{(\theta, \omega) \mid \theta = \omega\}$, when it is still possible to tell the exact time without knowing which hand is for hours and which for minutes). The reader can now easily find the number of these intersection points and compute that there are 143 of them in the intersection $\Gamma_1 \cap \Gamma_2$, and 11 in the intersection $\Gamma_1 \cap \Gamma_2 \cap \Delta$, which shows that there are all together 132 ambiguous positions.

REMARK 14.1.3

Let us note that the "watch with equal hands" problem reduces to counting points or 0-dimensional manifolds in the intersection of two circles, viewed as 1-dimensional submanifolds of the 2-dimensional manifold T^2. More generally, one may be interested in how many points there are in the intersection of two or more submanifolds of a higher-dimensional ambient manifold. Topology gives us a versatile tool for computing this and much more, in terms of the so-called *intersection product* $\alpha \frown \beta$ *of homology classes* α *and* β in a manifold M. This intersection product is, via Poincaré duality, equivalent to the "cup" product, and has the usual properties [Mun84]. In our Example 14.1.2, keeping in mind that $a \frown b = -\, b \frown a$ for all 1-dimensional classes, and in particular that $a \frown a = 0$ if $\dim\,(a) = 1$, we have $[\Gamma_1] \frown [\Gamma_2] = ([\theta] + 12[\omega]) \frown ([\omega] + 12[\theta]) = [\theta] \frown [\omega] + 12[\omega] \frown [\omega] + 12[\theta] \frown [\theta] + 144[\omega] \frown [\theta] = 143[\omega] \frown [\theta]$ and, taking the orientation into account, we conclude that the number of intersection points is 143.

14.2 PARTITIONS OF MASS DISTRIBUTIONS

Problems of partitioning mass distributions in the plane, 3-space, or spaces of higher dimension form the first circle of discrete geometric problems where topological methods have traditionally been applied with great success.

An (open) ham sandwich is a collection of three measurable sets in $\mathbb{R}^3$, representing a slice of bread, a slice of ham, and a slice of cheese. It turns out that there always exists a plane simultaneously halving all three measurable sets or, in other words, that a ham sandwich can be cut fairly into two pieces by a single straight cut. Suppose, on the other hand, that you want to split an irregularly shaped slice of pizza with a hungry friend who is supposed to divide the pizza into two pieces by a straight knife-cut, but who can cut anywhere he likes. You are allowed to mark your piece in advance by specifying a single point that will lie in your piece. Then, if you are very careful about marking your piece, you can count on at least one third of the pizza. These two results are instances of *the ham sandwich theorem* and *the center point theorem* which, together with their relatives, often require topological methods in their proofs.

GLOSSARY

Measure: An abstract function μ defined on a class of sets that has all the formal properties (additivity, positivity) of the usual *volume* or *area* functions.

Measurable set: Any set in the domain of the function μ.

Mass distribution and density function: A density function is an integrable function $f : \mathbb{R}^d \to [0, +\infty)$ representing the density of a "mass distribution" (measure) on $\mathbb{R}^d$. The measure μ arising this way is defined by $\mu(A) := \int_A f\, dx$.

Halving hyperplane: A hyperplane that simultaneously bisects a family of measurable sets.

Grassmann and ***Stiefel manifolds:*** The Grassmann manifold $G_k(\mathbb{R}^n)$ of all k-dimensional linear subspaces of $\mathbb{R}^n$ and the Stiefel manifold $V_k(\mathbb{R}^n)$ of all orthonormal k-frames in $\mathbb{R}^n$ are frequently used in the construction of configuration spaces associated to measure partitioning problems.

14.2.1 THE HAM SANDWICH THEOREM

Given a collection of d measurable sets (mass distributions, finite sets) in $\mathbb{R}^d$, the problem is to simultaneously bisect all of them by a single hyperplane. Often a measurable set is a geometric object $A \subset \mathbb{R}^d$, say a polytope, whose measure is simply its volume vol A. More generally, a measurable set A is an arbitrary subset of $\mathbb{R}^d$ if it is clear from the context what we mean by its "measure" $\mu(A)$. Typically, A is a Lebesgue-measurable set and $\mu(A) = m(A)$ its Lebesgue measure which, in the usual cases, reduces to the measure vol described above. More generally, if $f : \mathbb{R}^d \to \mathbb{R}^+$ is an integrable density function, then $\mu(A) := \int_A f\, dm = \int_{\mathbb{R}^d} f\phi_A\, dm$ is the measure or the mass distribution associated with the function f, where ϕ_A is the characteristic function of A (1 on A, 0 otherwise). An important special case arises if $f = \phi_B$ for a Lebesgue-measurable set B, where $\mu(A) = m(A \cap B)$. Finally, if $S \subset \mathbb{R}^d$ is a finite set, then $\mu(A) := |A \cap S|$ is the so-called ***counting measure*** induced by the set S. All of these examples are subsumed by the case of a positive, σ-additive Borel measure μ. This means that μ is defined on a σ-algebra $\mathcal{F}$ of subsets of $\mathbb{R}^d$ that includes all closed halfspaces and other sets that arise naturally in geometric problems. The reader should, in principle, not have any difficulty reformulating any of the following results for whatever special class of measures she may be interested in.

THEOREM 14.2.1 *Ham Sandwich Theorem*

Let $\mu_1, \mu_2, \ldots, \mu_d$ be a collection of measures (mass distributions, measurable sets, finite sets) in the sense above. Then there exists a hyperplane H such that for all $i = 1, \ldots, d$, $\mu_i(H^+) \geq 1/2\, \mu_i(\mathbb{R}^d)$ and $\mu_i(H^-) \geq 1/2\, \mu_i(\mathbb{R}^d)$, where H^+ and H^- are the closed halfspaces associated with the hyperplane H.

In the special case where $\mu(H) = 0$, i.e., where the hyperplane itself has measure zero, H is called a ***halving hyperplane*** since $\mu_i(H^+) = \mu_i(H^-) = 1/2\, \mu_i(\mathbb{R}^d)$ for all i. A halving hyperplane H is also called a "ham sandwich cut," for the reasons noted above.

TOPOLOGICAL BACKGROUND

The topological result lying behind the ham sandwich theorem is the Borsuk-Ulam theorem, [Ste85, Mata]. The proof of the ham sandwich theorem historically marks one of the first applications of the CS/TM-scheme, with the $(d-1)$-sphere as the configuration space, $\mathbb{R}^d$ as the test space, and $G = \mathbb{Z}_2$ as the group of symmetries associated to the problem. Given a collection $\{A_i\}_{i=1}^d$ of d measurable sets, the test map $t : S^{d-1} \rightarrow \mathbb{R}^d$ is defined by $t(e) = (\alpha_1, \ldots, \alpha_d)$, with α_i determined by the condition that $H_i := \{x \in \mathbb{R}^d \mid \langle x, e\rangle = \alpha_i\}$ is a halving hyperplane for the measurable set A_i. The test space is the diagonal $Z := \{(\alpha, \ldots, \alpha) \in \mathbb{R}^d \mid \alpha \in \mathbb{R}\}$. The test map t is obviously "odd", or $\mathbb{Z}_2$-equivariant, in the sense that $t(-e) = -t(e)$.

THEOREM 14.2.2 *Borsuk-Ulam Theorem*

For every continuous map $f : S^n \rightarrow \mathbb{R}^n$ from an n-dimensional sphere into n-dimensional Euclidean space, there exists a point $x \in S^n$ such that $f(x) = f(-x)$.

An important special case of the Borsuk-Ulam theorem arises if f is an odd map. The conclusion is that a continuous odd map must have a zero on the sphere, i.e., $f(x) = 0$ for some $x \in S^d$. This is precisely the reason why the test map t for the ham sandwich theorem has the property $t(e) \in Z$ for some $e \in S^{d-1}$. Note that the general Borsuk-Ulam theorem follows from the special case if the latter is applied to the map $\phi : S^d \rightarrow \mathbb{R}^d$ given by $\phi(x) = f(x) - f(-x)$.

There is a different topological approach to the ham sandwich theorem closer to the earlier example about a watch with two indistinguishable hands. Here we mention only that the role of the torus T^2 is played by a manifold M representing all hyperplanes in $\mathbb{R}^d$ (the configuration space), while the curves Γ_1 and Γ_2 are replaced by suitable submanifolds N_i of M, one for each of the measures μ_i, $i = 1, \ldots, d$. N_i is defined as the space of all halving hyperplanes for the measurable set A_i.

APPLICATIONS AND RELATED RESULTS

Let $S_1, \ldots, S_d$ be a collection of finite sets, called "colors," in $\mathbb{R}^d$. Assume that the size of each of these sets is n and that the points are all in general position. Then, according to Akiyama and Alon (see [Bár93]), the ham sandwich theorem implies that there exists a partition of $S := \bigcup_{i=1}^{d} S_i$ into n nonempty, pairwise disjoint sets $D_1, \ldots, D_n$ that are multicolored in the sense that $|D_i \cap S_j| = 1$ for all i and j, such that the simplices conv $D_1, \ldots,$ conv D_n are pairwise disjoint.

14.2.2 THE CENTER POINT THEOREM

THEOREM 14.2.3 *Center Point Theorem*

Let $A \subset \mathbb{R}^d$ be a Lebesgue-measurable subset of $\mathbb{R}^d$ or, more generally, one of the measures μ described prior to Theorem 14.2.1. Then there exists a point $x \in \mathbb{R}^d$ such that for every closed halfspace $P \subset \mathbb{R}^d$, if $x \in P$ then

$$\mathrm{vol}(P \cap A) \geq \frac{\mathrm{vol}(A)}{d+1}.$$

When formulated for a more general measure μ, the result guarantees that $\mu(P) \geq \mu(\mathbb{R}^d)/(d+1)$ for every closed halfspace $P \ni x$.

TOPOLOGICAL BACKGROUND

If the Borsuk-Ulam theorem is responsible for the ham sandwich theorem, then R. Rado's center point theorem can be seen as a consequence of another well-known topological result, Brouwer's fixed point theorem. Note that the usual formulation about self-maps $f : K \to K$ generalizes easily to the following formulation.

THEOREM 14.2.4 *Brouwer's Fixed Point Theorem*

Let K be a compact, convex body in $\mathbb{R}^n$. Suppose $f : K \to \mathbb{R}^n$ is a continuous map such that for each $x \in K$ the image $f(x)$ belongs to the supporting cone of K at x, $\mathrm{cone}_x(K) := \bigcup_{\lambda \geq 0}(x + \lambda(K - x))$. Then $f(x) = x$ for some $x \in K$.

Very often it is more convenient to use Kakutani's theorem, which is a generalization of Brouwer's theorem to "multivalued functions" $f : B \to R^n$.

The center point theorem is deduced from Brouwer's theorem roughly as follows. Let $x \in B$, where B is a "large" ball containing the set A. If x is not a center point, then there is a vector $e \in S^{d-1}$ pointing in a direction in which x can be moved to make it closer to being one. In this way we define a function $x \mapsto f(x)$, and a fixed point, i.e., a point that doesn't need to be moved, is a center point.

Recall that the center point theorem was originally deduced from Helly's theorem about intersecting families of convex sets, which also has several topological relatives.

APPLICATIONS AND RELATED RESULTS

The first proof of the center point theorem (R. Rado) was based on Helly's theorem. For this reason, it is often viewed as a measure-theoretic equivalent of Helly's theorem.

As noted by Miller and Thurston (see [MTTV97, MTTV98]), the center point theorem and the Koebe theorem on the disk representation of planar graphs can be used to prove the existence of a small separator for a planar graph, a result proved originally (by Lipton and Tarjan) by different methods.

The ***regression depth*** $\mathrm{rd}_{\mathcal{P}}(H)$ of a hyperplane H relative to a collection $\mathcal{P}$ of n points in $\mathbb{R}^d$ is the minimum number of points that H must pass through in moving to the vertical position. Dually, given an arrangement $\mathcal{H}$ of n hyperplanes in $\mathbb{R}^d$, the regression depth $\mathrm{rd}_{\mathcal{H}}(x)$ of a point x relative to $\mathcal{H}$ is the smallest k such that x cannot escape to infinity without crossing (or moving parallel to) at least k hyperplanes. The problem of finding a point (resp. hyperplane) with maximum regression depth relative to $\mathcal{H}$ (resp. $\mathcal{P}$) is shown in [AET00] to be intimately connected with the problem of finding center points. The main result (confirming a conjecture of Rousseeuw and Hubert) is that there always exists a point with regression depth $\lceil n/(d+1)\rceil$; cf. Chapter 57 of this Handbook.

14.2.3 CENTER TRANSVERSAL THEOREM

THEOREM 14.2.5 *Center Transversal Theorem*

Let $A_0, A_1, \ldots, A_k$, $0 \le k \le d-1$, be a collection of Lebesgue-measurable sets in $\mathbb{R}^d$ or, more generally, let $\mu_0, \mu_1, \ldots, \mu_k$ be a sequence of measures. Then there exists a k-dimensional affine subspace $D \subset \mathbb{R}^d$ such that for every closed halfspace $H(v,\alpha) := \{x \in \mathbb{R}^d \mid \langle x, v\rangle \le \alpha\}$ and every $i \in \{0,1,\ldots,k\}$,

$$D \subset H(v,\alpha) \Longrightarrow m(A_i \cap H(v,\alpha)) \ge \frac{m(A_i)}{d-k+1}.$$

If formulated for a sequence $\mu_0, \ldots, \mu_k$ of more general measures, the result guarantees that $\mu_i(H(v,\alpha)) \ge \mu_i(\mathbb{R}^d)/(d-k+1)$ for all i and all $H(v,\alpha) \supseteq D$.

TOPOLOGICAL BACKGROUND

The center transversal theorem contains the ham sandwich and center point theorems as two boundary cases [ZV90]. The topological principle that is at the root of this result should be strong enough for this purpose. This result has several incarnations. One of them, in the language of the CS/TM-scheme, is a theorem of E. Fadell and S. Husseini [FH88] that claims the nonexistence of a $\mathbb{Z}_2^{\oplus k}$-equivariant map $f : V_{n,k} \to (\mathbb{R}^k)^{n-k} \setminus \{0\}$ from the Stiefel manifold of all orthonormal k-frames in $\mathbb{R}^n$ to the sum of $n-k$ copies of $\mathbb{R}^k$. The group $\mathbb{Z}_2^{\oplus k}$ can be identified with the group of all diagonal matrices in $SO(k)$ and its action on $\mathbb{R}^k$ is induced by the obvious action of $SO(k)$. A related result [FH88, ZV90] is that the vector bundle $\xi_k^{\oplus(n-k)}$ does not admit a nonzero, continuous cross-section, where ξ_k is the tautological k-plane bundle over the Grassmann manifold $G_k(\mathbb{R}^n)$.

APPLICATIONS AND RELATED RESULTS

The following Helly-type transversal theorem, due to Dol'nikov (see [Eck93]), is a consequence of the same topological principle that is at the root of the center transversal theorem. Moreover, the center transversal theorem is related to Dol'nikov's result in the same way that the center point theorem is related to Helly's theorem.

THEOREM 14.2.6

Let $\mathcal{K}_0, \ldots, \mathcal{K}_k$ be families of compact convex sets. Suppose that for every i, and for each k-dimensional subspace $V \subset \mathbb{R}^d$, there exists a translate V_i of V intersecting every set in $\mathcal{K}_i$. Then there exists a common k-dimensional transversal of the family $\mathcal{K} := \bigcup_{i=0}^{k} \mathcal{K}_i$, i.e., there exists an affine k-dimensional subspace of $\mathbb{R}^d$ intersecting all the sets in $\mathcal{K}$.

Let $\mathcal{K} = \{K_0, ..., K_k\}$ be a family of convex bodies in $\mathbb{R}^n$, $1 \leq k \leq n-1$. Then an affine l-plane $A \subset \mathbb{R}^n$ is called a ***common maximal l-transversal*** of $\mathcal{K}$ if $m(K_i \cap A) \geq m(K_i \cap (A + x))$ for each $i \in \{0, ..., k\}$ and each $x \in \mathbb{R}^n$, where m is l-dimensional Lebesgue measure in A and $A + x$, respectively. It was shown in [MVZ01] that, given a family $\mathcal{K} = \{K_i\}_{i=0}^{k}$ of convex bodies in R^n $(k < l)$, the set $C_l(\mathcal{K})$ of all common maximal l-transversals of $\mathcal{K}$ has to be "large" from both the measure-theoretic and the topological point of view. Here again one uses the same topological principle responsible for all results in this section together with some integral geometry calculations to show that a cohomologically "big" subspace of the Grassmann manifold $G_k(\mathbb{R}^n)$ has to be large also in a measure-theoretic sense.

14.2.4 EQUIPARTITION OF MASSES BY HYPERPLANES

A measurable set $A \subset \mathbb{R}^3$ can be partitioned by three planes into 8 pieces of equal measure. This is an instance of the general problem of characterizing all triples (d, j, k) such that for any j mass distributions (measurable sets) in $\mathbb{R}^d$, there exist k hyperplanes, $k \leq d$, such that each of the 2^k "orthants" contains the fraction $1/2^k$ of each of the masses. Such a triple (d, j, k) will be called *admissible*. For example, the ham sandwich theorem implies that $(d, d, 1)$ is admissible. It is known (E. Ramos, [Ram96]) that $d \geq j(2^k - 1)/k$ is a necessary condition and $d \geq j2^{k-1}$ a sufficient one for a triple (d, j, k) to be admissible. Ramos's method yields many interesting results in lower dimensions, including the admissibility of the triples $(9, 3, 3)$, $(9, 5, 2)$, and $(5, 1, 4)$. The most interesting special case that still seems to be out of reach is the triple $(4, 1, 4)$. The key idea in these proofs is to use, for this purpose, a specially designed, generalized form of the Borsuk-Ulam theorem for continuous, "even-odd" maps of the form $f : S^{d-1} \times \ldots \times S^{d-1} \to \mathbb{R}^l$.

APPLICATIONS AND RELATED RESULTS

According to [Mata], an early interest of computer scientists in partitioning mass distributions by hyperplanes was stimulated in part by *geometric range searching*; cf. Chapter 36 of this Handbook. As noted by Matoušek, the classical mass partitioning results were eventually superseded by random sampling and related results. However, one still wonders about the possible impact of a positive answer to the following conjecture (a special case of the conjecture that $(4,1,4)$ is admissible) to the construction and complexity of geometric algorithms.

CONJECTURE 14.2.7

For each collection of 16 *distinct points* $A_1,\ldots,A_{16}$ *in* $\mathbb{R}^4$, *there exist* 4 *hyperplanes* $H_1,\ldots,H_4$ *such that each of the associated* 16 *open orthants contains at most one of the given points.*

It is known that the answer to the conjecture is positive if the points are distributed along a ***convex curve*** in $\mathbb{R}^4$ (a curve in $\mathbb{R}^m$ is convex if, like the moment curve, it intersects each hyperplane in at most m distinct points). This special case of the conjecture follows [Ram96] from the existence of uniform Gray codes on 4-dimensional cubes. Recall that a uniform Gray code on a k-dimensional cube is a Hamiltonian circuit on the graph of all edges of the cube that is balanced in the sense that it uses the same number of edges from each of k parallel classes.

14.2.5 RADIAL PARTITIONS BY POLYHEDRAL FANS

An old result of R. Buck and E. Buck [BB49] says that for each continuous mass distribution in the plane, there exist three concurrent lines $l_1,l_2,l_3\subset\mathbb{R}^2$ that partition $\mathbb{R}^2$ into six sectors of equal measure. It is natural to search for higher dimensional analogs of this result.

Suppose that $Q\subset\mathbb{R}^d$ is a convex polytope and assume that the origin $O\in\mathbb{R}^d$ belongs to the interior $\mathrm{int}(Q)$ of Q. Let $\{F_i\}_{i=1}^k$ be the collection of all facets of Q. Let $\mathcal{F}:=\mathrm{fan}(Q)$ be the associated ***fan***, i.e., $\mathcal{F}=\{C_1,\ldots,C_k\}$ where $C_i=\mathrm{cone}(F_i)$ is the convex closed cone with vertex O generated by F_i.

THEOREM 14.2.8 [Mak01]

Let Q *be a regular dodecahedron with the origin* $O\in\mathbb{R}^3$ *as its barycenter. Then for any centrally symmetric, continuous mass distribution* μ *on* $\mathbb{R}^3$, *there exists a linear map* $L\in GL(3,\mathbb{R})$ *such that*

$$\mu(L(C_1))=\mu(L(C_2))=\ldots=\mu(L(C_k)).$$

Makeev actually shows in [Mak01] that L can be found in the set of all matrices of the form $a\cdot t$, where t is an upper triangular matrix and $a\in GL(3,\mathbb{R})$ is a matrix given in advance. In an earlier paper (see [Mak98]) he showed that a radial partition by a fan determined by the facets of a cube always exists for an arbitrary measure in $\mathbb{R}^3$. Moreover, he shows in [Mak01] that a result analogous to Theorem 14.2.8 also holds for rhombic dodecahedra. Recall that the rhombic dodecahedron U_3 is the polytope bounded by twelve planes, each containing an edge of a cube and parallel to one of the great diagonal planes. A higher dimensional analogue of the rhombic dodecahedron is the polytope U_n in $\mathbb{R}^n$ described as the dual of the difference body of a regular simplex.

PROBLEM 14.2.9

Let $T\subset\mathbb{R}^n$ *be a regular simplex and* $Q:=T-T$ *the associated "difference polytope." Let* $U_n:=Q^\circ$ *be the polytope polar to* Q. *Clearly* U_n *is a centrally symmetric polytope with* n^2+n *facets* F_i, $i=1,\ldots,n^2+n$. *Let* $\{K_i\}_{i=1}^{n^2+n}$ *be the associated conical dissection of* $\mathbb{R}^n$, *where* $K_i:=\mathrm{cone}(F_i)$. *Is it true that for any continuous* *mass distribution μ on $\mathbb{R}^n$ there exists a nondegenerate affine map $A : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\mu(A(K_1)) = \mu(A(K_2)) = \ldots = \mu(A(K_{n^2+n}))\,?$$

The following result of Vrećica and Živaljević is an example of a radial partition result for a single measure in $\mathbb{R}^n$ with ratios prescribed by a positive vector α.

THEOREM 14.2.10 [VZ01]

Let $\Delta \subset \mathbb{R}^n$ be a nondegenerate simplex with $O \in \operatorname{int}(\Delta)$. Suppose that μ is a continuous mass distribution on $\mathbb{R}^n$, and let $\alpha = (\alpha_0, \ldots, \alpha_n)$ be a given positive vector such that $\alpha_0 + \ldots + \alpha_n = 1$. Then there exists a vector $v \in \mathbb{R}^n$ such that $\mu(v + K_i) = \alpha_i\, \mu(\mathbb{R}^n)$ for each $i = 0, \ldots, n$, where $\mathcal{F} = \operatorname{fan}(\Delta) = \{K_i\}_{i=0}^n$ is the radial fan associated to Δ.

14.2.6 EQUIPARTITIONS BY WEDGELIKE CONES

The center transversal theorem is a common generalization of the ham sandwich theorem and the center point theorem. There is another general statement extending the ham sandwich theorem that, as a special boundary case, includes the equipartition case of Theorem 14.2.10.

THEOREM 14.2.11 [VZ92]

Let $\Delta := \operatorname{conv}(\{a_i\}_{i=0}^m)$ be a regular simplex of dimension $m \leq d$ and let $P := \operatorname{aff} \Delta$ be its affine hull. Then there is a dissection $\mathcal{D}(\Delta) = \{D_i\}_{i=0}^m$ of $\mathbb{R}^d$ into $m+1$ wedgelike cones, where $D_i := P^\perp \oplus \operatorname{cone}(\operatorname{conv}(\{a_j\}_{j \neq i}))$.

CONJECTURE 14.2.12

Let $\mu_0, \ldots, \mu_k$ be a family of continuous mass distributions (measures), $0 \leq k \leq d-1$, defined on $\mathbb{R}^d$. Then there exists a $(d{-}k)$-dimensional regular simplex Δ such that for the corresponding dissection, $\mathcal{D}(\Delta)$, for some $x \in \mathbb{R}^d$, and for all i, j,

$$\mu_i(x + D_j) \geq \frac{\mu_i(\mathbb{R}^d)}{d-k+1}.$$

This conjecture is denoted in [VZ92] by $B(d,k)$. Theorem 14.2.10 implies $B(d,0)$, and the ham sandwich theorem is $B(d,d-1)$. The conjecture is also confirmed in the case $B(d,d-2)$ for all d. Moreover, there exists a natural topological conjecture implying $B(d,k)$ that is closely related to the analogous statement needed for the center transversal theorem. This statement, denoted in [VZ92] by $C(d,k)$, in the spirit of the CS/TM-scheme, essentially claims that there does not exists a $\mathbb{Z}_{k+1}$-equivariant map from the Stiefel manifold $V_k(\mathbb{R}^n)$ to the unit sphere $S(V)$ in an appropriate $\mathbb{Z}_{k+1}$-representation V.

14.2.7 PARTITIONS BY CONVEX SETS

CONJECTURE 14.2.13

Let n and d be integers with $n, d \geq 2$. Assume that $\mu_1, \ldots, \mu_d$ are continuous mass distributions such that $\mu_1(\mathbb{R}^d) = \ldots = \mu_d(\mathbb{R}^d) = n$. Then there exists a partition of

$\mathbb{R}^d$ *into* n *sets* $C_1, \ldots, C_n$ *such that the interiors* $\mathrm{int}(C_i)$ *are convex sets and that* $\mu_i(C_i) = 1$ *for each* $i = 1, \ldots, n$.

This conjecture was formulated in [KK99] by A. Kaneko and M. Kano for the case $d = 2$. Kaneko and Kano originally formulated the conjecture for finite sets rather than for continuous mass distributions, but this is not essential. Note that the case $n = 2$ is true by the ham sandwich theorem. The case $d = 2$ was independently established by S. Bespamyatnikh, D. Kirkpatrick, and J. Snoeyink, by T. Sakai, and by H. Ito, H. Uehara, and M. Yokoyama; see [BM01] for additional information.

14.2.8 PARTITIONS BY k-FANS IN PRESCRIBED RATIOS

The conjecture of Kaneko and Kano (the case $d = 2, n = 3$) motivated I. Bárány and J. Matoušek in [BM01, BM02] to study general conical partitions of planar or spherical measures in prescribed ratios. We assume, in the following statements, that all measures are continuous mass distributions.

An arrangement of k semilines in the Euclidean (projective) plane or on the 2-sphere is called a k-fan if all semilines start from the same point. A k-fan is an α-partition for a probability measure μ if $\mu(\sigma_i) = \alpha_i$ for each $i = 1, ..., k$, where $\{\sigma_i\}_{i=1}^k$ are conical sectors associated with the k-fan and $\alpha = (\alpha_1, ..., \alpha_k)$ is a given vector. The set of all $\alpha = (\alpha_1, ..., \alpha_m)$ such that for any collection of probability measures $\mu_1, ..., \mu_m$ there exists a common α-partition by a k-fan is denoted by $\mathcal{A}_{m,k}$. It was shown in [BM01] that the interesting cases of the problem of existence of α-partitions are $(k, m) = (2, 3), (3, 2), (4, 2)$.

CONJECTURE 14.2.14

Suppose that (k, m) *is equal to* $(2, 3), (3, 2)$ *or* $(4, 2)\}$. *Then* $\alpha \in \mathcal{A}_{k,m}$ *if and only if*

$$\alpha_1 + \ldots + \alpha_m = 1 \quad \text{and} \quad \alpha_i > 0 \quad \text{for each} \quad i = 1, \ldots m.$$

The only known elements in $\mathcal{A}_{4,2}$ are, up to a permutation of coordinates, $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})$. They were discovered by Bárány and Matoušek by an ingenious application of the CS/TM scheme [BM01, BM02]. From this Bárány and Matoušek deduced that $\{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})\} \cup \{(\frac{p}{5}, \frac{q}{5}, \frac{r}{5}) \mid p, q, r \in N^+, p + q + r = 5\} \subset \mathcal{A}_{3,2}$.

Conjecture 14.2.14 was confirmed in full in the case $(k, m) = (2, 3)$ by R. Živaljević in [Živ02]. Building on the CS/TM scheme of Bárány and Matoušek, he deduced the result from the fact that under mild conditions there does not exist a Q_{4n}-equivariant map $f : S^3 \to V \setminus \mathcal{A}(\alpha)$, where $\mathcal{A}(\alpha)$ is a Q_{4n}-invariant, linear subspace arrangement in a Q_{4n}-representation V, and Q_{4n} is the generalized quaternion group. This fact is in turn established by showing that an appropriate obstruction in the group $\Omega_1(Q_{4n})$ of Q_{4n}-bordisms does not vanish.

14.2.9 OTHER EQUIPARTITIONS

There are other types of partitions of mass distributions. A "cobweb partition theorem" of Schulman (see [Mata]) guarantees an equipartition of a plane mass distribution into 8 pieces by an arrangement of lines resembling a cobweb.

A result of Paterson (see [Mata]) is an interesting example of a ham sandwich-type theorem that deals with partitions of lines rather than of points. It says that for every set of lines in $\mathbb{R}^3$, there exist 3 mutually perpendicular planes such that the interior of each of the resulting octants is intersected by no more than half of the lines.

14.3 THE PROBLEMS OF BORSUK AND KNASTER

The topological methods used in proofs of measure partition results are actually applicable to a much wider class of combinatorial and geometric problems. This phenomenon can be partially explained by the fact that quite different problems, which on the surface have very little in common (say one of them may be discrete and the other not), may actually lead to the same or closely related configuration spaces and test maps. This in turn implies that such problems both follow from the same general topological principle and that they could, despite appearances, be classified as "relatives".

14.3.1 BORSUK'S PROBLEM

Borsuk's well-known problem about covering sets in $\mathbb{R}^n$ with sets of smaller diameter was solved in the negative by J. Kahn and G. Kalai [KK93] who proved that the size of a minimal cover is exponential in n; see Chapters 1 and 2 of this Handbook. This, however, gave a new impetus to the study of "Borsuk numbers" after the old exponential upper bounds suddenly became more plausible. This may be one of the reasons why results about "universal covers", originally used for these estimates, have received new attention in the last few years.

The following result was proved originally by V. Makeev; see also [HMS, Kup99]. Recall the rhombic dodecahedron U_3, the polytope bounded by twelve rhombic facets, which appeared in Section 14.2.5.

THEOREM 14.3.1 [Mak98]

A rhombic dodecahedron of width 1 *is a universal cover for all sets* $S \subset \mathbb{R}^3$ *of diameter* 1*. In other words, each set of diameter* 1 *in* 3*-space can be covered by a rhombic dodecahedron whose opposite faces are* 1 *unit apart.*

Let $\Sigma \subset \mathbb{R}^n$ be a regular simplex of edge-length 1, with vertices $v_1, \dots, v_{n+1}$. Then the intersection of $n(n+1)/2$ parallel strips S_{ij} of width 1, where S_{ij} is bounded by the $(n-1)$-planes orthogonal to the segment $[v_i, v_j]$ passing through the vertices v_i and v_j $(i < j)$, is a higher dimensional analog of the rhombic dodecahedron. It is easy to see that this is just another description of the polytope U_n that we encountered in Problem 14.2.9.

CONJECTURE 14.3.2 *Makeev conjecture* [HMS]

The polytope U_n *is a* universal cover *in* $\mathbb{R}^n$*. In other words, for each set* $S \subset \mathbb{R}^n$ *of diameter* 1*, there exists an isometry* $I : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $S \subset I(U_n)$*.*

The relevance of the Makeev conjecture for the general Borsuk problem is obvious. The following stronger conjecture is yet another example of a topological statement with potentially interesting consequences in discrete and computational geometry.

CONJECTURE 14.3.3 [HMS]
Let $f : S^{n-1} \to \mathbb{R}$ be an odd function, and let $\Sigma_n \subset \mathbb{R}^n$ be a regular simplex of edge-length 1, with vertices $v_1, \ldots, v_{n+1}$. Then there exists an orthogonal linear map $A \in SO(n)$ such that the $n(n+1)/2$ hyperplanes H_{ij}, $1 \le i < j \le n+1$, are concurrent, where

$$H_{ij} := \{x \in \mathbb{R}^n \mid \langle x, A(v_j - v_i)\rangle = f(A(v_j - v_i))\}.$$

G. Kuperberg showed in [Kup99] that, unlike the cases $n = 2$ and $n = 3$, for $n \ge 4$ there is homologically an even number of isometries $I : \mathbb{R}^n \to \mathbb{R}^n$ such that $S \subset I(U_n)$ for a given set S of constant width. Kuperberg showed that the Makeev conjecture can be reduced (essentially in the spirit of the CS/TM-scheme) to the question of the existence of a Γ-equivariant map $f : SO(n) \to V \setminus \{0\}$, where Γ is a group of symmetries of the root system of type A_n and the test space V is an $n(n-1)/2$-dimensional representation of Γ. The fact that such a map exists if and only if $n \ge 4$ may be an indication that the Makeev conjecture is false in higher dimensions.

14.3.2 KNASTER'S PROBLEM

Knaster's problem is one of the old conjectures of discrete geometry with a distinct topological flavor. The conjecture is now known to be false in general, but the problem remains open in many interesting special cases.

PROBLEM 14.3.4 *Knaster's problem* [Kna47]
Given a finite subset $S = \{s_1, \ldots, s_k\} \subset S^n$ of the n-sphere, determine the conditions on k and n so that for each continuous map $f : S^n \to \mathbb{R}^m$ there will exist an isometry $O \in SO(n+1)$ with

$$f(O(s_1)) = f(O(s_2)) = \ldots = f(O(s_k)).$$

Knaster originally conjectured that such an isometry O always exists if $k \le n - m + 2$. Just as in the case of the Borsuk problem, the first counterexamples took a long time to appear. V. Makeev, and somewhat later K. Babenko and S. Bogatyi (see [Che98]), showed that the condition $k \le n - m + 2$ is not sufficient if the original set S is sufficiently "flat." In [Che98], W. Chen constructed new counterexamples confirming that the (original) Knaster conjecture is false for all $n > m > 2$.

The fact that Knaster's conjecture is false in general does not rule out the possibility that for some special configurations $S \subset S^n$ the answer is still positive. The case where S is the set of vertices of a "big" regular simplex in S^n is of special interest since it directly generalizes the Borsuk-Ulam theorem.

Questions closely related to Knaster's conjecture are the problems of inscribing or circumscribing polyhedra to convex bodies in $\mathbb{R}^n$; see [HMS, Kup99]. G. Kuperberg observed that both the circumscription problem for constant-width bodies and Knaster's problem are special cases of the following problem.

PROBLEM 14.3.5 [Kup99]

Given a finite set T of points on S^{d-1} and a linear subspace L of the space of all functions from T to $\mathbb{R}^n$, decide if, for each continuous function $f : S^{d-1} \to \mathbb{R}^n$, there is an isometry O such that the restriction of $f \circ O$ to T is an element of L.

14.4 TVERBERG-TYPE THEOREMS AND THEIR APPLICATIONS

A collection of seven points in the plane can be partitioned into three nonempty, disjoint subsets so that the corresponding convex hulls have a nonempty intersection. If we add two more points and color all the points with three colors so that each color is equally represented, then there exists a partition of this set of nine colored points into three multicolored three-point sets such that the corresponding multicolored triangles have a nonempty intersection. Something similar is possible in 3-space, but this time we need five points of each color in order to guarantee a partition of this kind. In short, given a constellation of five blue, five red, and five yellow stars in space, it is always possible to form three vertex-disjoint multicolored triangles with nonempty intersection. These are the simplest nontrivial cases of Tverberg-type theorems, which, together with their consequences and most important applications, are shown in Figure 14.4.1.

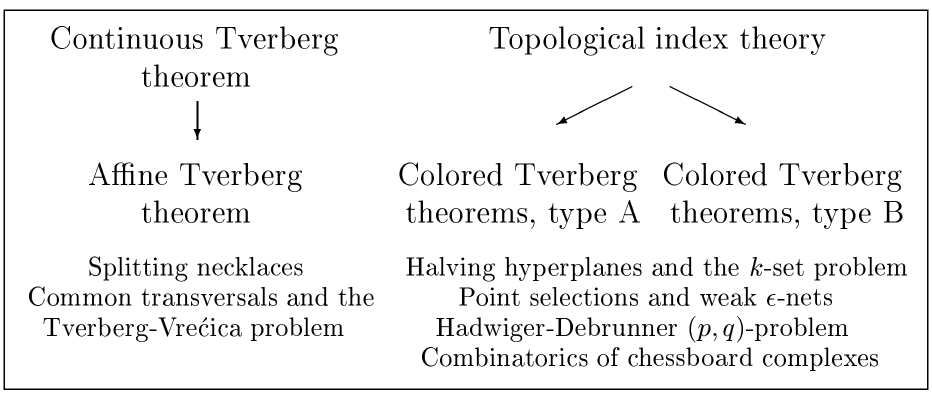

FIGURE 14.4.1
Tverberg-type theorems.

GLOSSARY

Tverberg-type problem: A problem in which a finite set $A \subset \mathbb{R}^d$ is to be partitioned into nonempty, disjoint pieces $A_1, \ldots, A_p$, possibly subject to some constraints, so that the corresponding convex hulls $\{\text{conv}(A_i)\}_{i=1}^p$ intersect.

Colors: A set of $k+1$ colors is a collection $\mathcal{C} = \{C_0, \ldots, C_k\}$ of disjoint subsets of $\mathbb{R}^d$, $d \geq k$. A set $B \subset \mathbb{R}^d$ is ***multicolored*** if it contains a point from each of the sets C_i; in this case conv B is called a ***rainbow simplex*** (possibly degenerate).

Type A and ***Type B:*** Colored Tverberg problems are of type A or type B depending on whether $k = d$ or $k < d$ (resp.), where $k + 1$ is the number of colors.

Tverberg numbers $T(r, d)$, $T(r, k, d)$: $T(r, k, d)$ is the minimal size of each of the colors C_i, $i = 0, \ldots, k$, that guarantees that there always exist r intersecting rainbow simplices. $T(r, d) := T(r, d, d)$.

14.4.1 MONOCHROMATIC TVERBERG THEOREMS

THEOREM 14.4.1 *Affine Tverberg Theorem*

Every set $K = \{a_j\}_{j=0}^{(q-1)(d+1)} \subset \mathbb{R}^d$ with $(d+1)(q-1)+1$ elements can be partitioned into q nonempty, disjoint subsets $K_1, \ldots, K_q$ so that the corresponding convex hulls have nonempty intersection:

$$\bigcap_{i=1}^{q} \operatorname{conv}(K_i) \neq \emptyset .$$

(The special case $q = 2$ is Radon's theorem; see Chapter 4.)

THEOREM 14.4.2 *Continuous Tverberg Theorem*

Let Δ^m be an m-dimensional simplex and assume that q is a prime integer. Then for every continuous map $f : \Delta^{(q-1)(d+1)} \to \mathbb{R}^d$ there exist vertex-disjoint faces $\Delta^{t_1}, \ldots, \Delta^{t_q} \subset \Delta^{(q-1)(d+1)}$ such that $\bigcap_{i=1}^q f(\Delta^{t_i}) \neq \emptyset$.

APPLICATIONS AND RELATED RESULTS

The affine Tverberg theorem was proved by Helge Tverberg in 1966. The continuous Tverberg theorem, proved by Bárány, Shlosman, and Szücs, reduces to the affine version if f is an affine (simplicial) map. It is not known if this result remains true for arbitrary q, although several authors have independently confirmed this if q is a prime power: see [Živ98] for a historical account. Some of the relevant references for these two theorems and their applications are [Bár93, Bjö95, Sar92, Eck93, Vol96, Živ98, Mat02, Mata].

The following "necklace-splitting theorem" of Noga Alon (see [Mata]) is a very nice application of the continuous Tverberg theorem.

THEOREM 14.4.3

Assume that an open necklace has ka_i beads of color i, $1 \leq i \leq t$, $k \geq 2$. Then it is possible to cut this necklace at $t(k-1)$ places and assemble the resulting intervals into k collections, each containing exactly a_i beads of color i.

REMARK 14.4.4

The proof of the necklace-splitting theorem provides a very nice example of an application of the CS/TM scheme (Section 14.1). A continuous model of a necklace is an interval $[0, 1]$ together with k measurable subsets $A_1, \ldots, A_k$ representing "beads" of different colors. It is well known that the configuration space of all sequences $0 \leq x_1 \leq \ldots \leq x_m \leq 1$ is the m-dimensional simplex, hence the totality of all m-cuts of a necklace is identified with an m-dimensional simplex Σ. Given a cut $c \in \Sigma$, the assembling of the resulting subintervals $I_0(c), \dots, I_m(c)$ of $[0,1]$ into k collections is determined by a function $f : [m+1] \to [k]$. Hence, a configuration space associated to the necklace-splitting problem is obtained by gluing together m-simplices Σ_f, one for each function $f \in \mathrm{Fun}([m+1],[k])$. The complex $\mathcal{C}_{m,k}$ obtained by this construction turns out, in fact, to be a very important example of a complex obtained from a simplex by a *deleted join operation*. The reader is refereed to [Mata] and [Živ98] for details about the role of (deleted) joins in combinatorics.

An interesting connection has emerged recently between ham-sandwich- and Tverberg-type problems. An example of this is the so-called Tverberg-Vrećica conjecture, which incorporates both the center transversal theorem (Theorem 14.2.5) and the (affine) Tverberg theorem in a single general statement.

CONJECTURE 14.4.5

Assume that $0 \le k \le d-1$ and let $S_0, S_1, \dots, S_k$ be a collection of finite sets in $\mathbb{R}^d$ of given cardinalities $|S_i| = (r_i - 1)(d-k+1)+1$, $i = 0,1,\dots,k$. Then S_i can be split into r_i nonempty sets, $S_i^1, \dots, S_i^{r_i}$, so that for some k-dimensional affine subspace $D \subset \mathbb{R}^d$, $D \cap \mathrm{conv}(S_i^j) \neq \emptyset$ for all i and j, $0 \le i \le k$, $1 \le j \le r_i$.

This conjecture was confirmed in [Živ99] for the case where both d and k are odd integers and $r_i = q$ for each i, where q is an odd prime number. Recently S. Vrećica confirmed this conjecture also in the case $r_1 = \ldots = r_k = 2$ [Vre02].

The expository article [Kal01] is recommended as a source of additional information about Tverberg-type theorems not covered here. From among Kalai's deep conjectures, beautiful visions, and unexpected possible connections (e.g. with the 4-color theorem), we select the following conjecture.

CONJECTURE 14.4.6 *Gil Kalai* (1974)

Given a set $A \subset \mathbb{R}^d$, let $T_r(A)$ be the set of all points in $\mathbb{R}^d$ that belong to the convex hull of r pairwise disjoint subsets of A. By convention let $\dim(\emptyset) = -1$. *Then*

$$\sum_{r=1}^{|A|} \dim(T_r(A)) \ge 0.$$

14.4.2 COLORED TVERBERG THEOREMS

Let $T(r,k,d)$ be the minimal number t so that for every collection of colors $\mathcal{C} = \{C_0, \dots, C_k\}$ with the property $|C_i| \ge t$ for all $i = 0, \dots, k$, there exist r multicolored sets $A_i = \{a_j^i\}_{j=0}^k$, $i = 1, \dots, r$, that are pairwise disjoint but where the corresponding rainbow simplices $\sigma_i := \mathrm{conv}\ A_i$ have a nonempty intersection, $\bigcap_{i=1}^r \sigma_i \neq \emptyset$.

The colored Tverberg problem is to establish the existence of, and then to evaluate or estimate, the integer $T = T(r,k,d)$. The cases $k = d$ and $k < d$ are related, but there is also an essential difference. In the case $k = d$, provided t is large enough, the number of intersecting rainbow simplices can be arbitrarily large. In the case $k < d$, for dimension reasons, one cannot expect more than $r \le d/(d-k)$ intersecting k-dimensional rainbow simplices. This is the reason why colored Tverberg theorems are classified as type A or type B, depending on whether

$k = d$ or $k < d$.

In the type A case, where $T(r, d, d)$ is abbreviated simply as $T(r, d)$, it is easy to see that a lower bound for this function is r. It is conjectured that this lower bound is attained:

CONJECTURE 14.4.7 (Type A)

$T(r, d) = r$ for all r and d.

This conjecture has been confirmed for $r = 2$ and for $d \leq 2$ [Bár93].

It is interesting to note (see Section 14.4.3) that the colored Tverberg problem (type A) was originally conjectured and designed as a tool for solving important problems of computational geometry. Note also that the weak form of the conjecture, $T(r, d) < +\infty$, is already far from obvious.

The following theorem of Živaljević and Vrećica (see [Bár93, Mata, Živ98]) provides the best bounds known in the general case. It implies that $T(r, d) \leq 4r - 3$ for all r and d.

THEOREM 14.4.8 (Type A)

For every integer r and every collection of $d+1$ disjoint sets ("colors") $C_0, C_1, \ldots, C_d$ in $\mathbb{R}^d$, each of cardinality at least $4r - 3$, there exist r disjoint, multicolored subsets $S_i \subset \bigcup_{i=0}^{d} C_i$ such that

$$\bigcap_{i=1}^{r} \operatorname{conv} S_i \neq \emptyset.$$

If r is a power of a prime number then it suffices to assume that the size of each of the colors is at least $2r - 1$. In other words, $T(r, d) \leq 2r - 1$ if r is a prime power and $T(r, d) \leq 4r - 3$ in the general case.

In the type B case, let us assume that $r \leq d/(d - k)$, which is a necessary condition for a colored Tverberg theorem of type B.

CONJECTURE 14.4.9 (Type B)

$T(r, k, d) = 2r - 1$.

There exist examples showing that $T(r, k, d) \geq 2r - 1$.

The following theorem [VZ94, Živ98] confirms Conjecture 14.4.9 above for the case of a prime power r.

THEOREM 14.4.10 (Type B)

Let $C_0, \ldots, C_k$ be a collection of $k + 1$ disjoint finite sets ("colors") in $\mathbb{R}^d$. Let r be a prime integer such that $r \leq d/(d - k)$ and let $|C_i| = t \geq 2r - 1$. Then there exist r multicolored k-dimensional simplices S_i, $i = 1, \ldots, r$, that are pairwise vertex-disjoint such that

$$\bigcap_{i=1}^{r} \operatorname{conv} S_i \neq \emptyset.$$

The usual price for using topological (equivariant) methods is the extra assumption that r is a prime or a power of a prime number. On the other hand, the results obtained by these methods hold in greater generality and include nonlinear versions of Theorems 14.4.8 and 14.4.10; see [Živ98] for details and examples.

EXAMPLE 14.4.11

The simplest instance of Theorem 14.4.10 *is the case* $d = 2$, $k = 1$, *and* $r = 2$. *Then, in the nonlinear version of this theorem, we recognize the well-known fact that the complete bipartite graph* $K_{3,3}$ *is not planar. This is one of the earliest results in topology, already known to Euler, who formulated it as a problem about three houses and three wells.*

14.4.3 APPLICATIONS OF COLORED TVERBERG THEOREMS

Theorem 14.4.8 provided a general bound of the form $T(d+1,d) \leq 4d+1$, which opened the possibility of proving many interesting results in discrete and computational geometry.

HALVING HYPERPLANES AND THE k-SET PROBLEM

The number $h_d(n)$ of halving hyperplanes of a set of size n in $\mathbb{R}^d$, i.e., the number of essentially distinct placements of a hyperplane that split the set in half, according to Bárány, Füredi, and Lovász (see [Bár93]), satisfies

$$h_d(n) = O(n^{d-\epsilon_d}), \quad \text{where} \quad \epsilon_d = T(d+1,d)^{-(d+1)}.$$

POINT SELECTIONS AND WEAK ϵ-NETS

The equivalence of the following statements was established in [ABFK92] before Theorem 14.4.8 was proved. Considerable progress has since been made in this area [Mat02], and different combinatorial techniques for proving these statements have emerged in the meantime.

- Weak colored Tverberg theorem: $T(d+1,d)$ is finite.
- Point selection theorem: There exists a constant $s = s_d$, whose value depends on the bound for $T(d+1,d)$, such that any family $\mathcal{H}$ of $(d{+}1)$-element subsets of a set $X \subset \mathbb{R}^d$ of size $|\mathcal{H}| = p\binom{|X|}{d+1}$ contains a pierceable subfamily $\mathcal{H}'$ such that $|\mathcal{H}'| \gg p^s\binom{|X|}{d+1}$. ($\mathcal{H}'$ is ***pierceable*** if $\bigcap_{S\in\mathcal{H}'} \operatorname{conv} S \neq \emptyset$. $A \gg_d B$ if $A \geq c_1(d)B + c_2(d)$, where $c_1(d) > 0$ and $c_2(d)$ are constants depending only on the dimension d.)
- Weak ϵ-net theorem: For any $X \subset \mathbb{R}^d$ there exists a weak ϵ-net F for convex sets with $|F| \ll_d \epsilon^{(d+1)(1-1/s)}$, where $s = s_d$ is as above. (See Chapter 36 for the notion of ϵ-net; a *weak* ϵ-net is similar, except that it need not be part of X.)
- Hitting set theorem: For every $\eta > 0$ and every $X \subset \mathbb{R}^d$ there exists a set $E \subset \mathbb{R}^d$ that misses at most $\eta\binom{|X|}{d+1}$ simplices of X and has size $|E| \ll_d \eta^{1-s_d}$, where s_d is as above.

OTHER RELATED RESULTS

A topological configuration space that arises via the CS/TM-scheme in proofs of Theorems 14.4.8 and 14.4.10 is the so-called ***chessboard complex*** $\Delta_{r,t}$, which owes its name to the fact that it can be described as the complex of all nontaking rook placements on an $r \times t$ chessboard. This is an interesting combinatorial object that arises independently as the coset complex of the symmetric group, as the complex of partial matchings in a complete bipartite graph, and as the complex of all partial injective functions. In light of the fact that the high connectivity of a configuration space is a property of central importance for applications (cf. Theorem 14.5.1), chessboard complexes have been studied from this point of view in numerous papers; see [Ath] and [Wac01] for recent advances and references.

14.5 TOOLS FROM EQUIVARIANT TOPOLOGY

The method of equivariant maps is a versatile tool for proving results in discrete geometry and combinatorics. For many results these are the only proofs available. Equivariant maps are typically encountered at the final stage of application of the CS/TM-scheme (Section 14.1).

GLOSSARY

G-space X, G-action: A group G acts on a space X if each element of G is a continuous transformation of X and multiplication in G corresponds to composition of transformations. Formally, a G-action α is a continuous map $\alpha : G \times X \to X$ such that $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$. Then X is called a G-space and $\alpha(g, x)$ is often abbreviated as $g \cdot x$ or gx.

Free G-action: An action is free if $g \cdot x = x$ for some $x \in X$ implies $g = e$, where e is the unit element in G.

G-equivariant map: A map $f : X \to Y$ of two G-spaces X and Y is equivariant if for all $g \in G$ and $x \in X$, $f(g \cdot x) = g \cdot f(x)$.

Borsuk-Ulam-type theorem: Any theorem establishing the nonexistence of a G-equivariant map between two G-spaces X and Y.

n-connected space: A path-connected and simply connected space with trivial homology in dimensions $1, 2, \ldots, n$. A path-connected space X is simply connected or 1-connected if every closed loop $\omega : S^1 \to X$ can be deformed to a point.

The following generalization of the Borsuk-Ulam theorem is the key result used in proofs of many Tverberg-type statements. Note that if $X = S^n$, $Y = S^{n-1}$, and $G = \mathbb{Z}_2$, it specializes to the "odd" form of the Borsuk-Ulam theorem given in Section 14.2 (following Theorem 14.2.2).

THEOREM 14.5.1

Suppose X and Y are simplicial (more generally CW) complexes equipped with the free action of a finite group G, and that X is m-connected, where $m = \dim Y$. Then there does not exist a G-equivariant map $f : X \to Y$.

Theorem 14.5.1 is strong enough for the majority of applications. Nevertheless, in some cases more sophisticated tools are needed. *A topological index theory* is a complexity theory for G-spaces that allows us to conclude that there does not exist a G-equivariant map $f : X \to Y$ if the G-space Y is of *larger complexity* than the G-space X. A measure of complexity of a given G-space is the so-called *equivariant* *index* $\mathrm{Ind}_G(X)$. In general, an index function is defined on a class of G-spaces, say all finite G-CW complexes, and takes values in a suitable partially ordered set Ω. For example if $G = \mathbb{Z}_2$, an index function $\mathrm{Ind}_{\mathbb{Z}_2}(X)$ is defined as the minimum integer n such that there exists a $\mathbb{Z}_2$-equivariant map $f : X \to S^n$. In this case $\Omega := \mathbb{N}$ is the poset of nonnegative integers. Note that the Borsuk-Ulam theorem simply states that $\mathrm{Ind}_{\mathbb{Z}_2}(S^n) = n$.

PROPOSITION 14.5.2 [Mata, Živ98]
For each nontrivial finite group G, there exists an integer-valued index function $\mathrm{Ind}_G(\cdot)$ defined on the class of finite, G-simplicial complexes such that

(i) *If* $\mathrm{Ind}_G(X) > \mathrm{Ind}_G(X)$, *then a G-equivariant map $f : X \to Y$ does not exist.*

(ii) *If X is $(n-1)$-connected then* $\mathrm{Ind}_G(X) \geq n$.

(iii) *If X is an n-dimensional, free G-complex then* $\mathrm{Ind}_G(X) \leq n$.

(iv) $\mathrm{Ind}_G(X * Y) \leq \mathrm{Ind}_G(X) + \mathrm{Ind}_G(Y) + 1$, *where $X * Y$ is the* join *of spaces.*

It is clear that the computation or good estimates of the complexity indices $\mathrm{Ind}_G(X)$ are essential for applications. Occasionally this can be done even if the details of construction of the index function are not known. Such a tool for finding the lower bounds for an index function described in Proposition 14.5.2 is provided by the following inequality.

PROPOSITION 14.5.3 *Sarkaria inequality* [Mata, Živ98]
Let L be a free G-complex and $L_0 \subset L$ a G-invariant, simplicial subcomplex. Let $\Delta(L \setminus L_0)$ be the order complex (cf. Chapter 21) *of the complementary poset $L \setminus L_0$. Then*

$$\mathrm{Ind}_G(L_0) \geq \mathrm{Ind}_G(L) - \mathrm{Ind}_G(\Delta(L \setminus L_0)) - 1.$$

In some applications it is more natural, and sometimes essential, to use more sophisticated partially ordered sets of G-degrees of complexity. A notable example is the *ideal valued index theory* of S. Husseini and E. Fadell [FH88], which proved useful in establishing the existence of equilibrium points in incomplete markets (mathematical economics).

14.6 SOURCES AND RELATED MATERIAL

FURTHER READING

The reader will find additional information about applications of topological methods in discrete geometry and combinatorics, as well as a more comprehensive bibliography, in the survey papers [Alo88, Bár93, Bjö95, Eck93, Ste85, Živ98] as well as in the books [Mat02, Mata].

The reader interested in broader aspects of the topology/computer science interaction is directed to the following sources:

(1) Both [BEA+99] and [DEG99], surveys of existing applications, may also be seen as programs offering an insight into future research in computational topology, identifying some of the most important general research themes in this field.

(2) The home page of the *BioGeometry project*, [BioG], also includes information (α-shapes, topological persistence, etc.) about the topological aspects of the problem of designing computational techniques and paradigms for representing, storing, searching, simulating, analyzing, and visualizing biological structures.

(3) The *CompuTop.org Software Archive* (maintained by Nathan Dunfield) is focused on software for low-dimensional topological computations [Dun].

(4) The Lisp computer program *Kenzo* [Ser] exemplifies the powerful computational techniques now available in *effective algebraic topology*.

(5) For general information about algebraic topology the reader may find the Web site [WD] of the Hopf Archive and the associated Topology discussion group (C. Wilkerson, D. Davis) extremely useful.

RELATED CHAPTERS

Chapter 1: Finite point configurations
Chapter 4: Helly-type theorems and geometric transversals
Chapter 32: Computational topology
Chapter 63: Biological applications of computational topology

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